Math 105 Notes
Real Analysis II

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Contents

Chapter 1

Lebesgue Measure and Integral

1.1 Motivation of Lebesgue Integral

Question. Given some subset $\Omega \subseteq \mathbb{R}^n$, and some real-valued function $f : \Omega \to \mathbb{R}$, is it possible to integrate f on Ω to obtain some number $\int_\Omega f$?

In one dimension we have the notion of a *Riemann integral* $\int_{[a,b]} f$, which answers the question when Ω is an interval $\Omega = [a.b]$ and f is *Riemann integrable.*

Remark. Note that every piecewise continuous function is Riemann integrable, and in particular every piecewise constant function is Riemann integrable. However, not all functions are Riemann integrable.

Although it is possible to extend such notion of a Riemann integral to higher dimensions, one can still only integrate "Riemann integrable" functions, which is a pretty small class of functions. (For example, the pointwise limit of Riemann integrable functions may not be Riemann integrable, but the uniform limits are.)
Thus, we need some truly satisfactory notion of integration that can handle even discontinuous functions, which leads us to the notion of the *Lebesgue integral*, which can handle a very large class of functions.

1.2 Lebesgue Measure

To understand how to compute an integral $\int_\Omega f$, we must know how to measure the length/area/volume of Ω. To understand the connection between the two, observe that if we integrate the function 1 on set Ω, then we obtain the length of Ω (if it's one-dimensional), the area of (if it's two-dimensional), the volume of Ω (if it's three-dimensional).

To avoid considering cases of the dimension, we will use the notion of *measure* $m(\Omega)$ to represent either length, area, volume (or hypervolumes, etc.) of Ω.

Question. Can we "consistently" define measure for any subset $\Omega \subseteq \mathbb{R}^n$?

Here are some desirable properties that we want:

(i) *Monotoncity*: if $A \subseteq B \subseteq \mathbb{R}^n$, then $m(A) \leq m(B)$.

(ii) *Additivity:* if $A \cap B = \emptyset$, then $m(A \cup B) = m(A) + m(B)$.

(iii) *Translational-invariance*: $m(x + A) = m(A)$ for any $x \in \mathbb{R}^n$, and $A \subseteq \mathbb{R}^n$.

Unfortunately, such a measure does not exist. It is impossible to define such a measure for *every* subset of $\mathbb{R}^n$, which goes against one's intuitive concept of volume (one interesting example of this failure of intuition is the *Banach-Tarski paradox*, in which a unit ball in $\mathbb{R}^3$ can be decomposed and reassembled to form two complete and disjoint unit balls via translations and rotations, thus doubling the volume.)

These paradoxes indicate that it is impossible to assign a measure to every single subset of $\mathbb{R}^n$, and so we can solve this issue by just simply consider measuring a certain class of sets in $\mathbb{R}^n$ called the *measurable sets*. By restricting our attention to these sets, we are able to define a measure with above properties.

1.2.1 Measurable Sets

Let $\mathbb{R}^n$ be a Euclidean space. For every measurable set $\Omega \subseteq \mathbb{R}^n$, we will define the *Lebesgue measure* $m(\Omega) \in [0, \infty]$. The measurable set will obey the following properties:

(i) *Borel property*: every open set in $\mathbb{R}^n$ is measurable, as is every closed set.

(ii) *Complementarity*: if Ω is measrable, then $\mathbb{R}^n \backslash \Omega$ is also measurable.

(iii) *Boolean algebra property*: if $(\Omega_j)_{j \in J}$ is any finite collection of measurable sets, then the union $\bigcup_{j \in J} \Omega_j$ and intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.

(iv) *σ-algebra property*: if $(\Omega_j)_{j \in J}$ are any countable collection of measurable sets, then the union $\bigcup_{j \in J} \Omega_j$ and intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.

For every measurable set Ω, we assign the *Lebesgue measure* $m(\Omega)$ that will satify the following properties:

(i) *Empty set*: $m(\emptyset) = 0$.

(ii) *Positivity*: $0 \leq m(\Omega) \leq +\infty$ for every measurable set Ω.

(iii) *Monotonicity*: if $A \subseteq B$, and A and B are both measurable, then $m(A) \leq m(B)$.

(iv) *Finite sub-additivity*: if $(A_j)_{j\in J}$ are a finite collection of measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) \leq \sum_{j\in J} m(A_j).$$

(v) *Finite additivity*: if $(A_j)_{j\in J}$ are a finite collection of disjoint measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} m(A_j).$$

(vi) *Countable sub-additivity*: if $(A_j)_{j\in J}$ are a countable collection of measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) \leq \sum_{j\in J} m(A_j).$$

(vii) *Countable additivity*: if $(A_j)_{j\in J}$ are a countable collection of disjoint measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} m(A_j).$$

(viii) *Normalization* The unit cube $[0,1]^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : 0 \leq x_j \leq 1 \text{ for all } 1 \leq j \leq n\}$ has measure $m([0,1]^n) = 1$.

(ix) *Translation invariance*: if Ω is a measurable set, and $x \in \mathbb{R}^n$, then $x + \Omega := \{x + y : y \in \Omega\}$ is also measurable, and $m(x + \Omega) = m(\Omega)$.

Theorem 1.2.1 (Existence of Lebesgue measure)**.** *There exists a concept of a measurable set, and a way to assign a number $m(\Omega)$ to every measurable subset $\Omega \subseteq \mathbb{R}^n$, which obeys all of the properties above.*

Remark. Lebesgue measure is pretty much unique. However, there are other measures which would only obey some of the above axioms, and we may be interested in measures for other domains than Euclidean spaces. This leads to the subject of *measure theory*.

1.2.2 Outer measure

Before we construct Lebesgue measure, we first discuss a naive approach to find the measure of a set by covering the set with boxes and then add up the volume of each box. This leads to the notion of *outer measure* which can be applied to every set and obeys most of the properties except for the additivity properties. We would need to modify it slightly later to recover the additivity property.

Definition 1.2.2 (Open box)**.** An *open box* $B \subseteq \mathbb{R}^n$ is any set of the form

$$B = \prod_{i=1}^{n} (a_i, b_i),$$

where $b_i \geq a_i$ are real numbers. We define the *volume* $\operatorname{vol}(B)$ of such box to be the number

$$\operatorname{vol}(B) := \prod_{i=1}^{n} (b_i - a_i).$$

Remark. Open boxes are open in general dimension.

Definition 1.2.3 (Covering by boxes)**.** Let $\Omega \subseteq \mathbb{R}^n$. A collection $(B_j)_{j \in J}$ of boxes *cover* Ω iff $\Omega \subseteq \bigcup_{j \in J} B_j$.

Suppose $\Omega \subseteq \mathbb{R}^n$ can be covered by a finite or countable collection of boxes $(B_j)_{j \in J}$. If we wish Ω to be measurable and have a measure obeying the monotonicity and sub-additivity properties, and $m(B_j) = \operatorname{vol}(B_j)$ for every box j, then we must have

$$m(\Omega) \leq m\left(\bigcup_{j \in J} B_j\right) \leq \sum_{j \in J} m(B_j) = \sum_{j \in J} \operatorname{vol}(B_j).$$

Thus, we conclude

$$m(\Omega) \leq \inf \left\{ \sum_{j \in J} \operatorname{vol}(B_j) : \bigcup_{j \in J} B_j \supseteq \Omega; J \text{ at most countable} \right\}.$$

Then we have our formal definition of the outer measure.

Definition 1.2.4 (Outer measure)**.** If Ω is a set, we define the *outer measure* $m^*(\Omega)$ of Ω to be the quantity

$$m^*(\Omega) := \inf \left\{ \sum_{j \in J} \operatorname{vol}(B_j) : \bigcup_{j \in J} B_j \supseteq \Omega; J \text{ at most countable} \right\}.$$

Remark. Outer measure can be defined for every single set (not just the measurable ones) because we can take the infimum of any non-empty set.

Lemma 1.2.5 (Properties of outer measure)**.**

(i) (Empty set) The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.

(ii) (Positivity) $0 \leq m^*(\Omega) \leq +\infty$ *for every measurable set* Ω.

(iii) (Monotonicity) If $A \subseteq B \subseteq \mathbb{R}^n$, *then* $m^*(A) \leq m^*(B)$.

(iv) (Finite sub-additivity) If $(A_j)_{j \in J}$ *are a finite collection of subsets of* $\mathbb{R}^n$, *then*

$$m^*\left(\bigcup_{j \in J} A_j\right) \leq \sum_{j \in J} m^*(A_j).$$

(v) (Countable sub-additivity) If $(A_j)_{j \in J}$ *are a countable collection of subsets of* $\mathbb{R}^n$, *then*

$$m^*\left(\bigcup_{j \in J} A_j\right) \leq \sum_{j \in J} m^*(A_j).$$

(vi) (Translation invariance) If Ω *is a subset of* $\mathbb{R}^n$, *and* $x \in \mathbb{R}^n$, *then* $m^*(x+\Omega) = m^*(\Omega)$.

Proof.

(i) This is obvious. Every open box covers the empty set.

(ii) Since we are taking the infimum over non-negative numbers, the result must be non-negative, and possibly infinite.

(iii) This is obvious. Every open covering of B must also be an open covering of A. Also if $M, N \subseteq \mathbb{R}$ with $N \subseteq M$, we have $\inf M \leq \inf N$. Thus, $m^*(A) \leq m^*(B)$.

(iv) For $n = 2$. We want to show that $m^*(A \cup B) \leq m^*(A) + m^*(B)$ by showing $m^*(A) + m^*(B) \geq m^*(A \cup B) - \epsilon$. For any $\epsilon > 0$, there exists a covering $(B_j)_j$ such that $\sum_j \mathrm{vol}(B_j) \leq m^*(A) + \epsilon$. Similarly for B. Then take the union of the two countable covers to get a cover of $A \cup B$.

(v) We want to show that for any $\epsilon > 0$, there exists a collection of open covers $(B_j^{(i)})_j$ for A_i such that

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_i m^*(A_i) + \epsilon = \sum_{i=1}^{\infty} \left(m^*(A_i) + \frac{\epsilon}{2^i}\right).$$

We can find open cover $(B_j^{(i)})$ for A_i such that

$$m^*(A_i) + \frac{\epsilon}{2^i} \geq \sum_{j=1}^{\infty} \mathrm{vol}(B_j^{(i)})$$

and

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} \mathrm{vol}(B_j^{(i)})\right) \geq m^*\left(\bigcup_{i=1}^{\infty} A_i\right).$$

(vi) If $(B_j)_j$ covers Ω, then $(x + B_j)_j$ covers $x + \Omega$.

□

Proposition 1.2.6 (Outer measure of closed box)**.** *For any closed box* $B = \prod_{i=1}^n [a_i, b_i]$*, we have*

$$m^*(B) = \prod_{i=1}^n (b_i - a_i) = vol(B).$$

Proof. We can cover the closed box $B = \prod_{i=1}^n [a_i, b_i]$ by the open box $\prod_{i=1}^n (a_i - \epsilon, b_i + \epsilon)$ for every $\epsilon > 0$. Thus

$$m^*(B) \leq \text{vol}\left(\prod_{i=1}^n (a_i - \epsilon, b_i + \epsilon)\right) = \prod_{i=1}^n (b_i - a_i + 2\epsilon).$$

Letting $\epsilon \to 0$ yields

$$m^*(B) \leq \prod_{i=1}^n (b_i - a_i) = \text{vol}(B).$$

Now we need to show that

$$m^*(B) \geq \text{vol}(B),$$

but by the definition of $m^*(B)$, it suffices to show that

$$\sum_{j \in J} \text{vol}(B_j) \geq \text{vol}(B)$$

if $(B_j)_{j \in J}$ is a finite or countable cover of B.

Since B is closed and bounded, it is compact (by Heine-Borel theorem), and thus every open cover of B has a finite subcover. Thus it suffices to prove the above inequality for finite covers (since if $(B_j)_{j \in J'}$ is a finite subcover of $(B_j)_{j \in J}$ then $\sum_{j \in J} \text{vol}(B_j) \geq \sum_{j \in J'} \text{vol}(B_j)$).

First we consider the base case $n = 1$. Then $B = [a, b]$ is just a closed interval and each box $B_j = (a_j, b_j)$ is an open interval. We need to show that

$$\sum_{j \in J} (b_j - a_j) \geq b - a.$$

Doing so requires Riemann integral. Let $f_j : \mathbb{R} \to \mathbb{R}$ be the indicator function such that $f_j(x) = \mathbf{1}_{B_j}(x)$, i.e. $f_j(x) = 1$ whenever $x \in B_j$ and 0 otherwise. Since f_j is piecewise constant, it is Riemann integrable and

$$\int_{\mathbb{R}} f_j(x) dx = b_j - a_j.$$

Summing this over all $j \in J$ gives

$$\int_{\mathbb{R}} f_j(x)dx = \sum_{j\in J} b_j - a_j.$$

Interchanging the integral with the finite sum (since we have a finite subcover), we obtain

$$\int_{\mathbb{R}} \sum_{j\in J} f_j(x)dx = \sum_{j\in J}(b_j - a_j).$$

But since $B \subseteq \bigcup_{j\in J} B_j$, we have

$$\mathbf{1}_B(x) \leq \sum_{j\in J} \mathbf{1}_{B_j}(x) = \sum_{j\in J} f_j(x)$$

and so

$$\int_{\mathbb{R}} \sum_{j\in J} f_j(x)dx \geq \int_{\mathbb{R}} \mathbf{1}_B(x)dx = b - a = \mathrm{vol}(B).$$

Now assume inductively that $n > 1$ and that the claim holds for dimensions $n-1$. We now have each box $B^{(j)}$ to be of the form

$$B^{(j)} = \prod_{i=1}^{n} (a_i^{(j)}, b_i^{(j)}).$$

We can write this as

$$B^{(j)} = A^{(j)} \times (a_n^{(j)}, b_n^{(j)})$$

where $A^{(j)}$ is the $(n-1)$-dimensional box of the form

$$A^{(j)} \coloneqq \prod_{i=1}^{n-1} (a_i^{(j)}, b_i^{(j)}).$$

Note that

$$\mathrm{vol}(B_j) = \mathrm{vol}_{n-1} A^{(j)} (b_n^{(j)} - a_n^{(j)})$$

where the subscript $n-1$ indicates that this is a $(n-1)$-dimensional volume. We similarly write

$$B = A \times [a_n, b_n]$$

where $A \coloneqq \prod_{i=1}^{n-1} [a_i, b_i]$ is the $(n-1)$-dimensional closed box. Thus, we also have

$$\mathrm{vol}(B) = \mathrm{vol}_{n-1} A(b_n - a_n).$$

Now for define $f^{(j)}(x_n) = \mathrm{vol}_{n-1}(A^{(j)})\mathbf{1}_{(a_n^{(j)},b_n^{(j)})}$.Then $f^{(j)}$ is Riemann integrable and

$$\int_{-\infty}^{\infty} f^{(j)} = \mathrm{vol}_{n-1}(A^{(j)})(b_n^{(j)} - a_n^{(j)}) = \mathrm{vol}(B^{(j)})$$

and thus

$$\sum_{j\in J} \mathrm{vol}(B^{(j)}) = \int_{-\infty}^{\infty} \sum_{j\in J} f^{(j)}.$$

Now let $x_n \in [a_n, b_n]$ and $(x_1, \dots, x_{n-1}) \in A$. Then $(x_1, \dots, x_n)$ lies in B and thus lies in oe of $B^{(j)}$. Clearly we have $x_n \in (a_n^{(j)}, b_n^{(j)}$, and $(x_1, \dots, x_{n-1}) \in A^{(j)}$. In particular, we see that for each $x_n \in [a_n, b_n]$, the set

$$\{A^{(j)} : j \in J; x_n \in (a_n^{(j)}, b_n^{(j)})\}$$

of $(n-1)$-dimensional boxes covers A. Then by inductive hypothesis, we see that

$$\sum_{j\in J : x_n \in (a_n^{(j)}, b_n^{(j)})} \mathrm{vol}_{n-1}(A^{(j)}) \geq \mathrm{vol}_{n-1}(A).$$

In other words,

$$\sum_{j\in J} f^{(j)}(x_n) \geq \mathrm{vol}_{n-1}(A).$$

Integrating this over $[a_n, b_n]$, we obtain

$$\int_{[a_n,b_n]} \sum_{j\in J} f^{(j)} \geq \mathrm{vol}_{n-1}(A)(b_n - a_n) = \mathrm{vol}(B).$$

In particular,

$$\int_{-\infty}^{\infty} \sum_{j\in J} f^{(j)} \geq \mathrm{vol}_{n-1}(A)(b_n - a_n) = \mathrm{vol}(B),$$

since $\sum_{j\in J} f^{(j)}$ is always non-negative. Combining this with our previous identity for $\int_{-\infty}^{\infty} \sum_{j\in J} f^{(j)}$ we obtain our claim, and the induction is complete. □

Corollary 1.2.7 (Outer measure of open box)**.** *For any open box $B = \prod_{i=1}^n (a_i, b_i)$, we have*

$$m^*(B) = \prod_{i=1}^n (b_i - a_i) = \mathit{vol}(B).$$

Proof. Assume that $b_i > a_i$ for all i since if $b_i = a_i$, we simply have an empty set and $m^*(\emptyset) = 0$. Now note that

$$\prod_{i=1}^n [a_i + \epsilon, b_i - \epsilon] \subseteq \prod_{i=1}^n (a_i, b_i) \subseteq \prod_{i=1}^n [a_i, b_i]$$

for all $\epsilon > 0$, assuming that ϵ is small enough that $b_i - \epsilon > a_i + \epsilon$ for all i. Then by the properties of the outer measure and the previous proposition, we have

$$\prod_{i=1}^{n}(b_i - a_i - 2\epsilon) \leq m^*\left(\prod_{i=1}^{n}(a_i, b_i)\right) \leq \prod_{i=1}^{n}(b_i - a_i).$$

Letting $\epsilon \to 0$ and using squeeze theorem completes our proof. □

Corollary 1.2.8 (Outer measure of any box)**.** *The outer measure of any box is equal to the volume of the box.*

Now let's see some examples of outer measure on $\mathbb{R}$.

Example 1.2.9 (Outer measure of $\mathbb{R}$)**.** Since $(-R, R) \subseteq \mathbb{R}$ for all $R > 0$, we have

$$m^*(\mathbb{R}) \geq m^*((-R, R)) = 2R.$$

Letting $R \to +\infty$ gives us $m^*(\mathbb{R}) = +\infty$.

Example 1.2.10 (Outer measure of $\mathbb{Q}$)**.** Since $\mathbb{Q}$ is countable, we can express $\mathbb{Q}$ as a union of all rational points q:

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}.$$

Then

$$m^*(\mathbb{Q}) = m^*\left(\bigcup_{q \in \mathbb{Q}} \{q\}\right) \leq \sum_{q \in \mathbb{Q}} m^*(\{q\}) = \sum_{q \in \mathbb{Q}} 0 = 0.$$

Thus, $m^*(\mathbb{Q}) = 0$, which is quite fascinating.

Remark. The same argument implies that every countable set has measure zero. (This also gives another proof that the real numbers are uncountable.)

Example 1.2.11 (Outer measure of $\mathbb{R}\backslash\mathbb{Q}$)**.** By finite-subadditivity, we have

$$m^*(\mathbb{R}) \leq m^*(\mathbb{R}\backslash\mathbb{Q}) + m^*(\mathbb{Q}).$$

Since $m^*(\mathbb{Q}) = 0$ and $m^*(\mathbb{R}) + \infty$, $m^*(\mathbb{R}\backslash\mathbb{Q}) = +\infty$ as expected.

Example 1.2.12 (Outer measure of $\mathbb{N}$)**.** By countable sub-additivity, we have

$$m^*(\mathbb{N}) \leq \sum_{n=0}^{\infty} m^*(\{n\}) = \sum_{n=0}^{\infty} 0 = 0.$$

1.2.3 Outer measure is not additive

Proposition 1.2.13 (Failure of countable additivity)**.** *There exists a countable collection $(A_j)_{j\in J}$ of disjoint subsets of $\mathbb{R}$, such that*

$$m^*\left(\bigcup_{j\in J} A_j\right) \neq \sum_{j\in J} m^*(A_j).$$

Proof. We will need the concept of *coset*. We say that $A \subseteq \mathbb{R}$ is a coset of $\mathbb{Q}$ if $A = x + \mathbb{Q}$ for some $x \in \mathbb{R}$. Note that any two cosets are either identical or distinct, and thus cannot partially overlap.

Observe that every coset $A = x + \mathbb{Q}$ for some $x \in \mathbb{R}$ has a non-empty intersection with $[0, 1]$. To see this, pick a rational number $q \in [-x, 1-x]$, then we have $x + q \in [0, 1]$, and so $x + q \in A \cap [0, 1]$.

Let $\mathbb{R}\backslash\mathbb{Q}$ denote the set of all cosets of $\mathbb{Q}$. For each coset A in $\mathbb{R}\backslash\mathbb{Q}$, pick an element $x_A \in A \cap [0, 1]$. (Note that this requires us to make an infinite number of choices, and thus requires the *axiom of choice*). Now let $E = \{x_A : A \in \mathbb{R}\backslash\mathbb{Q}\}$. Note that $E \subseteq [0, 1]$ by construction. Consider the set

$$X = \bigcup_{q\in\mathbb{Q}\cap[-1,1]} (q + E).$$

Clearly this is contained in $[-1, 2]$ (since $q + x \in [-1, 2]$ whenever $q \in [-1, 1]$ and $x \in E \subseteq [0, 1]$). We claim that this set contains $[0, 1]$. In fact, for any $y \in [0, 1]$, we know that y must belong to some coset A (such as $A = y + \mathbb{Q}$). But also have x_A belonging to the same coset, and so $y - x_A$ is equal to some rational q. Since y and x_A are both in $[0, 1]$, then q is in $[-1, 1]$. Since $y = q + x_A$, we have $y \in q + E$, and thus $y \in X$ as desired.

Now we claim that

$$m^*(X) \neq \sum_{q\in\mathbb{Q}\cap[-1,1]} m^*(q + E),$$

which would prove the claim. To see this, observe that since $[0, 1] \subseteq X \subseteq [-1, 2]$, that we have $1 \leq m^*(X) \leq 3$ by monotonicity and previous proposition. For the right hand side, by translation invariance that

$$\sum_{q\in\mathbb{Q}\cap[-1,1]} m^*(q + E) = \sum_{q\in\mathbb{Q}\cap[-1,1]} m^*(E).$$

The set $\mathbb{Q}\cap[-1, 1]$ is countably infinite. Thus, the right hand side is either 0 (if $m^*(E) = 0$) or $+\infty$ (if $m^*(E) > 0$), and neither lies between 1 and 3, and we are done. □

Proposition 1.2.14 (Failure of finite additivity). *There exists a finite collection* $(A_j)_{j \in J}$ *of disjoint subsets of* $\mathbb{R}$*, such that*

$$m^*\left(\bigcup_{j \in J} A_j\right) \neq \sum_{j \in J} m^*(A_j).$$

Proof. Exercise. □

1.3 Measurable sets

Definition 1.3.1 (Lebesgue measurability). Let $E \subseteq \mathbb{R}^n$. We say that E is *Lebesgue measurable*, or *measurable* for short, iff we have

$$m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$$

for every subset $A \subseteq \mathbb{R}^n$. If E is measurable, we define the *Lebesgue measure* of E to be $m(E) = m^*(E)$; if E is not measurable, we leave $m(E)$ undefined.

In other words, E is measurable if we can use it to divide up an arbitrary set A into two parts, the additivity property holds. We can think of the measurable sets as the sets for which finite additivity works.

We rarely verify measurability of a set with the above definition. Instead, we will prove some useful properties that will come in handy in the future.

Lemma 1.3.2 (Half-spaces are measurable). *The half-space*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

is measurable.

Proof. For $n = 1$: we want to show for all $A \subseteq \mathbb{R}$ that

$$m^*(A) = m^*(A_+) + m^*(A_-)$$

where $A_+ = A \cap (0, \infty)$, $A_- = A \cap (-\infty, 0]$. Since $A = A_+ \sqcup A_-$, by sub-additivity,

$$m^*(A) \leq m^*(A_+) + m^*(A_-).$$

To show $m^*(A) \geq m^*(A_+) + m^*(A_-)$, it suffice to show for all $\epsilon > 0$,

$$m^*(A) + \epsilon \geq m^*(A_+) + m^*(A_-).$$

Consider an open cover of A by $(B_j)_j$ open boxes such that

$$\sum_j \mathrm{vol}(B_j) \le m^*(A) + \frac{\epsilon}{2}.$$

Define

$$B_j^+ = B_j \cap (0, \infty), \qquad B_j^- = B_j \cap (-\infty, \epsilon/2^{j+1}).$$

Then $B_j = B_j^- \cup B_j^+$, and

$$\mathrm{vol}(B_j) + \frac{\epsilon}{2^{j+1}} \ge \mathrm{vol}(B_j^+) + \mathrm{vol}(B_j^-) \ge \mathrm{vol}(B_j).$$

Since $\bigcup B_j^+ \supseteq A_+$ and $\bigcup B_j^- \supseteq A_-$, we have

$$\begin{aligned} m^*(A_+) + m^*(A_-) &\le \sum_j \mathrm{vol}(B_j^+) + \sum_j \mathrm{vol}(B_j^-) \\ &\le \sum_j \left(\mathrm{vol}(B_j) + \frac{\epsilon}{2^{j+1}}\right) \\ &\le \sum_j \mathrm{vol}(B_j) + \frac{\epsilon}{2} \\ &\le m^*(A) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= m^*(A) + \epsilon. \end{aligned}$$

□

Remark. Any half-space of similar form is measurable.

Lemma 1.3.3 (Properties of measurable sets)**.**

(i) If E is measurable, then $\mathbb{R}^n \backslash E$ is also measurable.

(ii) If E is measurable and $x \in \mathbb{R}^n$, then $x + E$ is also measurable, and $m(x+E) = m(E)$.

(iii) If E_1 and E_2 are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.

(iv) If $E_1, E_2, \ldots, E_N$ are measurable, then $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are measurable.

(v) Every open box, and every closed box, is measurable.

(vi) Any set E of outer measure zero is measurable.

Proof.

(i) We want to show that $\forall A \subseteq \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap E^c) + m^*(A \backslash E^c).$$

Note that $A \cap E^c = A \backslash E$ and $A \backslash E^c = A \cap E$. Thus we need to show

$$m^*(A) = m^*(A \backslash E) + m^*(A \cap E),$$

which is already true by the measurability of E.

(ii) Since outer measure is translation-invariant,

$$m^*(A) = m^*(A \cap (x+E)) + m^*(A \backslash (x+E))$$

is equivalent to

$$m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x) \backslash E),$$

which is true by measurability of E.

(iii) We want to show that $\forall A \subseteq \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \backslash (E_1 \cap E_2)).$$

Define $A_{++} = A \cap E_1 \cap E_2$, $A_{+-} = A \cap E_1 \cap E_2^c$, $A_{-+} = A \cap E_1^c \cap E_2$, and $A_{--} = A \cap E_1^c \cap E_2^c$. Then

$$A = A_{++} \sqcup A_{+-} \sqcup A_{-+} \sqcup A_{--}.$$

Then what we want to show now is that

$$m^*(A) = m^*(A_{++}) + m^*(A \backslash A_{++}).$$

We can show using E_1 being measurable that

$$m^*(A) = m^*(A_{+-} \cup A_{++}) + m^*(A_{-+} \cup A_{--})$$

and

$$m^*(A) = m^*(A_{-+} \cup A_{--}) + m^*(A_{--} \cup A_{-+})$$

since E_2 is measurable. The rest is left as an exercise.

(iv) Use induction on number of operands.

(v) Boxes are intersections of (translated) half-spaces, i.e.

$$[a,b] = [a,\infty) \cap (-\infty,b] = (a+[0,\infty)) \cap (b+(-\infty,0]).$$

Then by measurability of half-spaces and translation invariance we proved above, boxes are measurable.

(vi) We need to show that for any $A \subseteq \mathbb{R}^n$

$$m^*(A) = m^*(A \cap E) + m^*(A \backslash E).$$

By sub-additivity, we have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \backslash E).$$

Now we need to show that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \backslash E).$$

By monotonicity, we have $m^*(A \cap E) \leq m^*(E) = 0$, so $m^*(A \cap E) = 0$. Now we only need to show

$$m^*(A) \geq m^*(A \backslash E),$$

which is true by monotonicity.

□

Lemma 1.3.4 (Finite additivity)**.** *If $(E_j)_{j \in J}$ are a finite collection of disjoint measurable sets and any set A (not necessarily measurable), we have*

$$m^*\left(A \cap \bigcup_{j \in J} E_j\right) = \sum_{j \in J} m^*(A \cap E_j).$$

Furthermore, we have

$$m\left(\bigcup_{j \in J} E_j\right) = \sum_{j \in J} m(E_j).$$

Proof. Exercise. □

Corollary 1.3.5. *If $A \subseteq B$ are two measurable sets, then $B \backslash A$ is also measurable, and*

$$m(B \backslash A) = m(B) - m(A).$$

Proof. Exercise. □

Lemma 1.3.6 (Countable additivity)**.** *If $(E_j)_{j \in J}$ are a countable collection of disjoint measurable sets, then $\bigcup_{j \in J} E_j$ is measurable, and*

$$m\left(\bigcup_{j \in J} E_j\right) = \sum_{j \in J} m(E_j).$$

Proof. Exercise. □

Lemma 1.3.7 (σ-algebra property). *If $(\Omega_j)_{j \in J}$ are any countable collection of measurable sets (so J is countable), then $\bigcup_{j \in J} \Omega_j$ and $\bigcap_{j \in J} \Omega_j$ are also measurable.*

Proof. Exercise. □

Lemma 1.3.8. *Every open set can be written as a countable or finite union of open boxes.*

Proof. Todo. □

Lemma 1.3.9 (Borel property). *Every open set, and every clsoed set, is Lebesgue measurable.*

Proof. It suffices to prove this for open sets, since the claim follows for closed sets by Lemma 1.3.3 (i). Let E be an open set. Then by Lemma 1.3.8, E is the countable union of open boxes. Since these open boxes are measurable, and that the countable union of measurable sets is measurable, E must be measurable. □

1.4 Measurable functions