STOCHASTIC PROCESSES

STAT 150

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Contents

1 Probability Review

1.1 Basic Definitions

Definition 1.1.1 (Probability Space). A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a triple consisting of a set Ω called the *sample space*, a set $\mathcal{F} \subseteq \Omega$ satisfying certain closure properties, and a function $\mathbb{P} : \mathcal{F} \to [0,1]$ that assigns probabilities to events in a coherent way.

Requirements for $\mathcal{F}$:

(i) $\Omega \in \mathcal{F}$.

(ii) If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$.

(iii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$, then

$$\bigcap_{i=1}^{\infty} E_i \in \mathcal{F}.$$

Requirements for $\mathbb{P}$:

(i) $\mathbb{P}(\Omega) = 1$.

(ii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint (meaning $E_i \cap E_j = \varnothing$ for $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Definition 1.1.2 (Random Variable). A *random variable* is a function $X : \Omega \to \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ whenever B is a "nice" subset of $\mathbb{R}$.

Example 1.1.3. $\Omega = \{H, T\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(\{H\}) = \frac{1}{2}$. $X(H) = 1, \quad X(T) = 0$.

$$\mathbb{P}(X = 1) = \mathbb{P}(\{H\}) = \frac{1}{2}, \quad \mathbb{P}(X = 0) = \mathbb{P}(\{T\}) = \frac{1}{2}.$$

1.2 Overview

Definition 1.2.1 (Stochastic Process). A *stochastic process* is a collection $\{X_t : t \in T\}$ of random variables $X_t : \Omega \to S \subseteq \mathbb{R}$ all defined on the some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here T is some index set (typically representing time) and S is the *state space*. One write this as

$X : \Omega \times T \to S$, $(w, t) \mapsto X_t(\omega)$. For a given outcome $\omega \in \Omega$, we get a sample path trajectory $X(\omega) : T \to S, t \mapsto X_t(\omega)$. A stochastic process can then be thought of as a random function.

The theme of this course is what can we say about the distribution of trajectories?

Example 1.2.2 (Branching Process (DTDS)). $X_0 = 1$, one individual in the 0th generation individuals produce a random number of offspring, i.i.d. $(\xi_i^{(n)})_{i\in\mathbb{N},n\in\mathbb{N}_0}$.

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

One interesting question would be what is $\mathbb{P}(X_n = 0 \text{ eventually})$, the probability of dying out?

Example 1.2.3 (Poisson Process (CTDS)). Recall that the Poisson distribution is used to model the number of occurrences of a rare event in some fixed period of time. The Poisson process $(N_t)_{t\geq 0}$ models the number of occurrences throughout time. $N_t = \#$ of occurrences by time t.

1.3 Useful Properties

(i) *(DeMorgan)*
$$(E \cup F)^c = E^c \cap F^c, \quad (E \cap F)^c = E^c \cup F^c.$$

(ii) *(Complementation)*
$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c).$$

(iii) *(Inclusion-exclusion)*
$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$
$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{S\in[n]:|S|=j} \mathbb{P}\left(\bigcap_{i\in S} E_i\right).$$

(iv) *(Partitioning)* If $\bigsqcup_{i=1}^{\infty} E_i = \Omega$, then
$$\mathbb{P}(F) = \mathbb{P}\left(\bigsqcup_{i=1}^{\infty} (F \cap E_i)\right) = \sum_{i=1}^{\infty} \mathbb{P}(F \cap E_i)$$

1.4 Conditional Probability

Conditioning: For $\mathbb{P}(F) > 0$,
$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

$\mathbb{P}(\cdot \mid F)$ defines a new probability measure on $(\Omega, \mathcal{F})$.

Multiplication rule:
$$\mathbb{P}(E \cap F) = \mathbb{P}(F)\mathbb{P}(E \mid F).$$

If $\bigsqcup_{i=1}^{\infty} F_i = \Omega$, then

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E \cap F_i) = \sum_{i=1}^{\infty} \mathbb{P}(F_i)\mathbb{P}(E \mid F_i).$$

Bayes' rule:

$$\mathbb{P}(F_j \mid E) = \frac{\mathbb{P}(F_j)\mathbb{P}(E \mid F_j)}{\sum_{i=1}^{\infty} \mathbb{P}(F_i)\mathbb{P}(E \mid F_i)}$$

1.5 Random Variables

1.5.1 Discrete Random Variables

If $X : \Omega \to S \subseteq \mathbb{R}$ is discrete,

$$\mathbb{P}(X \in E) = \sum_{x \in E} \mathbb{P}(X = x) = \sum_{x \in E \cap S} \mathbb{P}(X = x).$$

1.5.2 Indicator Random Variable

$$X(\omega) = \mathbb{I}_E(\omega) = \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E. \end{cases}$$

1.5.2.1 Binomial Random Variable

$$X = \sum_{i=1}^{n} \mathbb{I}_{E_i}, \quad \mathbb{P}(E_i) = p.$$

$$p_X(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

1.5.3 Continuous Random Variables

If X continuous,

$$\mathbb{P}(X \in E) = \int_E f_X(x) dx.$$

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x) dx.$$

1.5.3.1 Exponential Random Variable

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{x \geq 0}.$$

1.5.3.2 Gaussian Random Variable

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

1.5.4 Cumulative Distribution Function (CDF)

$F_X : \mathbb{R} \to [0, 1]$,

$$F_X(r) = \mathbb{P}(X \leq r) = \mathbb{P}(X \in (-\infty, r]).$$

If X is discrete,

$$F_X(r) = \sum_{x_i \leq r} p_X(x_i).$$

If X is continuous,

$$F_X(r) = \int_{-\infty}^{r} f_X(x)dx.$$

$$\frac{d}{dr} F_X(r) = f_X(r).$$

1.5.5 Expectation

1.5.5.1 Discrete case

$$\mathbb{E}[X] = \sum_{x_i \in S} x_i \mathbb{P}(X = x_i).$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

$$\mathbb{E}[g(X)] = \sum_{x_i \in S} g(x_i)\mathbb{P}(X = x_i).$$

1.5.5.2 Continuous case

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx.$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\mathbb{P}(X \geq x)dx.$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

1.5.6 Variance

$$\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

1.5.7 Moments

$$\mathbb{E}[X^m] = \int_0^\infty m x^{m-1} \mathbb{P}(X \geq x) dx.$$

1.5.8 Joint Distribution

1.5.8.1 Discrete

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

1.5.8.2 Continuous

$$\mathbb{P}((X,Y) \in E) = \int \int_E f_{X,Y}(x,y) dx dy$$

1.5.8.3 Marginal Distribution

$$p_X(x) = \sum_{y \in S_Y} p_{X,Y}(x,y).$$

$$f_X(x) = \int_{y \in S_y} f_{X,Y}(x,y) dy$$

1.5.9 Independence

$$p_{X,Y}(x,y) = p_X(x) p_Y(y).$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

$$\mathbb{P}(X \leq x, Y \leq y) = F_X(x) F_Y(y).$$

1.5.10 Linearity of Expectation

$$\mathbb{E}\left[\sum_{i=1}^n c_i X_i\right] = \sum_{i=1}^n c_i \mathbb{E}[X_i]$$

If $(X_i)_{i=1}^n$ independent,

$$(g(X_i))_{i=1}^n$$

independent.

$$\mathbb{E}\left[\prod_{i=1}^n g(X_i)\right] = \prod_{i=1}^n \mathbb{E}\left[g(x_i)\right]$$

$$\operatorname{Var}\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n \operatorname{Var}(x_i)$$

In general,

$$\operatorname{Var}\left(\sum_{i=1}^{n} x_i\right) = \sum_{i,j=1}^{n} \operatorname{Cov}(x_i, x_j)$$

1.5.11 Convolution

Discrete case: X, Y discrete $X \perp\!\!\!\perp Y$

$$\begin{aligned}
\mathbb{P}(X+Y=z) &= \sum_{Y} \mathbb{P}(X+Y=z, Y=y) \\
&= \sum_{y} \mathbb{P}(X=z-y, Y=y) \\
&= \sum_{y} \mathbb{P}(X=z-y)\mathbb{P}(Y=y) \quad \left(=\sum_{x} \mathbb{P}(X=x)\mathbb{P}(Y=z-x)\right).
\end{aligned}$$

If X, Y are $\mathbb{Z}$-valued, this becomes

$$\begin{aligned}
\mathbb{P}(X+Y=n) &= \sum_{k=-\infty}^{\infty} \mathbb{P}(X=n-k)\mathbb{P}(Y=y) \\
&= \sum_{h=-\infty}^{\infty} \mathbb{P}(X=k)\mathbb{P}(Y=n-k) \\
&= (\mathbb{P}_X * \mathbb{P}_Y)(n)
\end{aligned}$$

Example 1.5.1 (Poisson). $X \sim$ Poisson (λ), $Y \sim$ Poisson (μ), $X+Y \sim$ Poisson $(\lambda+\mu)$

$$\begin{aligned}
\mathbb{P}(X+Y=n) &= \sum_{h=0}^{n} \mathbb{P}(X=k)P(Y=n-k) \\
&= \sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{\mu!}e^{-\mu}\frac{\mu^{n-\mu}}{n-k!} \\
&= e^{-(\lambda+\mu)} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\lambda^k \mu^{n-k} \\
&= \frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^n \\
&= \mathbb{P}(Z=n)
\end{aligned}$$

where $Z \sim \text{Poisson}(\lambda+\mu)$.

Continuous case: X, Y continuous

$$\begin{aligned}
\mathbb{P}(X+Y \leqslant z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_X(x)f_Y(y)dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{z} f_X(x)f_Y(y-x)dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x)f_Y(y-x)dxdy
\end{aligned}$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx = \int_{-\infty}^{\infty} f_Y(y)f_X(z-y)dy = f_X * f_Y.$$

Example 1.5.2 (Convolution in uniform distributions). $X, Y \sim U[0,1]$, $X \perp\!\!\!\perp Y$.

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx.$$

$$f_X(x) = \mathbb{I}_{[0,1]}(x) \qquad f_Y(y) = \mathbb{I}_{[0,1]}(y)$$

so

$$\begin{aligned} f_{X+Y}(z) &= \int_{x\in[0,1], z-x\in[0,1]} 1dx \\ &= \int_{x\in[0,1], x\in[-1+z,z]} 1dx \\ &= \int_{\max(0,-1+z)}^{\min(1,z)} 1dx \\ &= \min(1,z) - \max(0,-1+z). \end{aligned}$$

1.5.12 Gamma Distribution

Definition 1.5.3 (Gamma function). Let $\alpha > 0$. The *gamma function* $\Gamma : (0,\infty) \to (0,\infty)$ is defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx = \mathbb{E}[X^{\alpha-1}]$$

where $X \sim \text{Exp}(1)$ Let $\alpha, \lambda > 0$. The $\text{Gamma}(\alpha, \lambda)$ distribution is defined by

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} \mathbb{I}_{x\geq 0}.$$

Exercise 1.5.4. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. (Hint: use induction)

1.5.13 Moment Generating Function

Definition 1.5.5 (MGF). For a random variable X, the *moment generating function* (MGF) is the function $M_X : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$,

$$M_X(t) = \mathbb{E}[e^{tX}].$$

If $M_X(t) < +\infty$ for $t \in (-\epsilon, \epsilon)$, then

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^k \mathbb{E}[x^k]}{k!} \text{ for } |t| < \epsilon$$

For independent RVs $(X_i)_{i=1}^n$,

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$$

Exercise 1.5.6. If $X \sim \text{Exp}(\lambda)$, then $M_x(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda, +\infty$ otherwise.
If $X \sim \text{Gamma}(n, \lambda)$, then

$$M_x(t) = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

If $X \sim \text{Gamma}(\alpha, \lambda)$, then

$$M_x(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}.$$

1.6 Conditional Probability (Cont'd)

Exercise 1.6.1 (Generalization). $(X_i)_{i=1}^n, (Y_j)_{j=1}^m$

$$p_{X_1,\ldots,X_n|Y_1,\ldots,Y_m}(x_1,\ldots,x_n \mid y_1,\ldots,y_m) = ?$$

Example 1.6.2. Let $M \in \mathbb{N}$ and $p, q \in (0,1)$. Consider $N \sim \mathrm{Bin}(M,q)$ and $X \sim \mathrm{Bin}(N,p)$. What is the distribution of X?

$$\begin{aligned}
\mathbb{P}(X=k) &= \sum_{n=0}^{M} \mathbb{P}(N=n)\mathbb{P}(X=k \mid N=n) \\
&= \sum_{n=0}^{M} \binom{M}{n} q^n (1-n)^{M-n} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{p^k}{k!} \sum_{n=k}^{M} \frac{M!}{(M-n)!(n-k)!} q^n (1-q)^{M-n} (1-p)^{n-k} \\
&= \frac{p^k}{k!(M-k)!} \sum_{n=k}^{M} \frac{M!(M-k)!}{(M-n)!(n-k)!} q^n (1-q)^{M-n} (1-p)^{n-k} \\
&= \binom{M}{k} p^k q^k \sum_{n=k}^{M} \binom{M-k}{n-k} q^{n-k} (1-q)^{M-j} (1-p)^{n-k} \\
&= \binom{M}{k} p^k q^k \sum_{t=0}^{M-k} \binom{M-k}{t} (q(1-p))^t (1-q)^{M-k-t} \\
&= \binom{M}{k} (pq)^k (q(1-p) + (1-q))^{M-k} \\
&= \binom{M}{k} (pq)^k (1-pq)^{M-k}.
\end{aligned}$$

Thus, $X \sim \mathrm{Bin}(M, pq)$.

Remark. What if $k > n$ in $\mathbb{P}(X=k \mid N=n)$ above in the first line? The probability is simply 0.

Question. Why does this answer make sense?

Answer. Think about retesting whenever we succeeded for the first M trials. Then X is simply the number of trials with double successes, thus we have the pq parameter.

Exercise 1.6.3. Consider $N \sim \mathrm{Poisson}(\lambda), X \sim \mathrm{Bin}(N,p)$. What is the distribution of X?

Answer. $X \sim \mathrm{Poisson}(\lambda p)$.

Question. How can we interpret this?

Answer. We can interpret X as the number of customers visiting a store who purchase something.

1.6.1 Conditional Expectation

For X, Y discrete, $g : \mathbb{R} \to \mathbb{R}$. Assume $\mathbb{E}[|g(X)|] = \sum_x |g(x)p_X(x) < \infty$.

Definition 1.6.4 (Conditional expectation)**.** The *conditional expectation* is defined as

$$\mathbb{E}[g(X) = y] = \sum_x g(x)p_{X|Y}(x|y)$$

if $p_Y(y) \neq 0$.

Remark. Note that $\mathbb{E}[g(X) \mid Y = y]$ is a real number, whereas $\mathbb{E}[g(X) \mid Y]$ is a random variable.

1.6.1.1 Tower Property

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[g(X) \mid Y]] &= \mathbb{E}\left[\sum_y \mathbb{E}[g(X) \mid Y = y]\right] \\
&= \sum_y \mathbb{E}[g(X) \mid Y = y]p_Y(y) \\
&= \sum_y \sum_x g(x)p_{X|Y}(x|y)p_Y(y) \\
&= \sum_x g(x) \sum_y p_{X|Y}(x|y)p_Y(y) \\
&= \sum_x g(x)p_X(x) \\
&= \mathbb{E}[g(X)].
\end{aligned}$$

Remark. One intuitive example would be considering the averages of heights of students from a classroom. We divide it into several groups and let Y denote the whichever group we select and let $\mathbb{E}[g(X) \mid Y]$ be the average of those from group Y. Then the average height of the entire classroom $\mathbb{E}[g(X)]$ is equivalent to the average of the average of heights of each group, which is $\mathbb{E}[\mathbb{E}[g(X) \mid Y]]$.

Properties of conditional expectations:

1. $\mathbb{E}\left[c_1 g\left(x_1\right) + c_2 h\left(x_2\right) \mid Y = y\right] = c_1 E\left[g\left(X_1\right) \mid Y = y\right] + c_2 E\left[h\left(X_2\right) \mid Y = y\right]$
2. If $g \geq 0$, then $E[g(x) \mid Y = y] \geq 0$.
3. $\mathbb{E}[f(X, Y) \mid Y = y] = \mathbb{E}[f(X, y) \mid Y = y]$.
4. If $X \perp\!\!\!\perp Y, \mathbb{E}[g(X) \mid Y = y] = \mathbb{E}[g(X)]$
5. $\mathbb{E}[g(x)h(y) \mid Y = y] = h(y)\mathbb{E}[g(x) \mid Y = y]$
6. $\mathbb{E}[g(x)h(y)] = \sum_y h(y)E[g(x) \mid Y = y]p_y(y) = \mathbb{E}[h(Y)\mathbb{E}[g(X) \mid Y]]$

Proof of 3.

$$\begin{aligned}\mathbb{E}[f(X,Y) \mid Y=y] &= \sum_{x,z} f(x,z)p_{X,Y|Y}(x,z|y) \\ &= \sum_{x,z} f(x,z)\frac{p_{X,Y,Y}(x,z,y)}{p_Y(y)} \\ &= \sum_{x} f(x,y)\frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \mathbb{E}[f(X,y) \mid Y=y].\end{aligned}$$

□

Remark. $\mathbb{E}[f(X,y)] \neq \mathbb{E}[f(X,y) \mid Y=y]$.

2 Random Sums

Definition 2.0.1. Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d random variables, N be a $\mathbb{N}_0$-valued random variable, $N \perp\!\!\!\perp (\xi_i)_{i=1}^{\infty}$. The *random sum* is defined as

$$X = \sum_{i=1}^{N} \xi_i = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} \xi_i \right) \mathbf{1}_{N=n} = \begin{cases} \sum_{i=1}^{n} \xi_i & \text{if } N = n \geq 1 \\ 0 & \text{if } N = 0. \end{cases}$$

Question. What is the distribution of X?

Let X, N be random variables. N is $\mathbb{N}_0$-valued. The condition CDF is

$$F_{X|N}(x|n) = \mathbb{P}(X \leq x \mid N = n)$$

if $\mathbb{P}(N = n) \neq 0$. The is an actual CDF, but for the random variable $X \mid N = n$.

Suppose that X is continuous and $F_{X|N}(x|n)$ is a differentiable function of x for each n such that $p_N(n) > 0$. The conditional PDF is

$$f_{X|N}(x|n) = \frac{d}{dx} F_{X|N}(x|n).$$

$$\begin{aligned} \int_a^b f_{X|N}(x|n) dx &= F_{X|N}(b|n) - F_{X|N}(a|n) \\ &= \mathbb{P}(X \in [a, b] \mid N = n). \end{aligned}$$

Answer.

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n) \mathbb{P}(X \leq x \mid N = n).$$

$$f_X(x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n) f_{X|N}(x|n).$$

2.1 Mean and Variance of Random Sums

Assume $\mathbb{E}[N] = \nu$ and $\mathbb{E}[\xi_i] = \mu$. Then

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid N]] \\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N} \xi_i \mid N\right]\right] \\
&= \mathbb{E}[N\mathbb{E}[\xi_1]] \\
&= \mathbb{E}[N\mu] \\
&= \mu\nu.
\end{aligned}$$

3 Markov Chains

3.1 Discrete-time Markov Chains

Definition 3.1.1 (Markov process)**.** A is a stochastic process $(X_t)_{t\in T}$ such that the future, given the present, is independent of the past.

Definition 3.1.2 (Markov property)**.** The *Markov property* for a DTDS stochastic process is

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}\left(X_{n+1} = j \mid X_n = i\right)$$

Example 3.1.3 (Gambler's ruin)**.** $(X_n)_{n=0}^{\infty}, X_n =$ your wealth after n turns. Stop if $X_n = 0$ or 5. Each play, you win \$1 with probability p and lose \$1 with probability $1-p$ independently of all previous plays. This process satisfies the markov property.

Example 3.1.4 (Ehrenfest model)**.** Box of N particles. $X_n =$ number of particles on the left side at time n. $N - X_n$ be the number of particles on the other side.

$$\mathbb{P}(X_{n+1} = i+1 \mid X_n = i) = \frac{N-i}{N}$$
$$\mathbb{P}(X_{n+1} = i-1 \mid X_n = i) = \frac{i}{N}.$$

Theorem 3.1.5.
Joint PMF of the Markov Chain is determined by initial distribution and $P = (p_{i,j})_{i,j\in S}$.

Proof.

$$\begin{aligned}\mathbb{P}(X_n = i_n, \dots, X_0 = i_0) &= \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0)\mathbb{P}(X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\ &= p_{i_{n-1},i_n} p_{i_{n-2},i_{n-1}} \cdots p_{i_0,i_1} \mathbb{P}(X_0 = i_0).\end{aligned}$$

□

3.1.1 n-step transition probabilities

$$p_{i,j} = \mathbb{P}(X_{n+1} = j \mid X_n = i).$$

Theorem 3.1.6.

$$p_{i,j}^{(m)} = \mathbb{P}(X_{n+m} = j \mid X_n = i) = (P^m)_{i,j}.$$

Proof.

$$\begin{aligned}\mathbb{P}(X_{n+m+1} = j \mid X_n = i) &= \sum_{k\in S} \mathbb{P}(X_{n+m+1} = j, X_{n+m} = k \mid X_n = i) \\ &= \sum_{k\in S} \mathbb{P}(X_{n+m+1} = j \mid X_{n+m} = k)\mathbb{P}(X_{n+m} = k \mid X_n = i).\end{aligned}$$

□

Example 3.1.7.

$$\begin{aligned}p_{i,j}^{(2)} &= \mathbb{P}(X_2 = j \mid X_0 = i) \\ &= \sum_{k\in S} \mathbb{P}(X_2 = j, X_1 = k \mid X_0 = i) \\ &= \sum_{k\in S} \mathbb{P}(X_2 = j \mid X_1 = k, X_0 = i)\mathbb{P}(X_1 = k \mid X_0 = i) \\ &= \sum_{k\in S} P_{i,k}P_{k,j} \\ &= (P^2)_{i,j}\end{aligned}$$

Example 3.1.8 (Inventory model)**.** X_n = inventory that you have of this product after the nth business day. If $X_n \le s$, place an order that brings inventory back to S by next morning. ξ_n = demand on nth day and (ξ_n) are i.i.d..

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases}\mathbb{P}(\xi_{n+1} = S - j) & \text{if } i \le s \\ \mathbb{P}(\xi_{n+1} = i - j) & \text{if } i > s.\end{cases}$$

$\lim_{n\to\infty} \mathbb{P}(X_n < 0)$ = chance of excess demand.

3.2 First Step Analysis

Consider $(X_n)_{n\ge 0}$ Markov chain on $\{1, \dots, r\} \cup \{r+1, \dots, N\}$ where $\{1, \dots, r\}$ are the *transient states* and $\{r + 1, \dots, N\}$ are the *absorbing states* such that

$$\begin{aligned}\lim_{n\to\infty} p_{i,j}^{(n)} &= 0 \qquad \forall i, j \in \{1, \dots, r\} \\ \lim_{n\to\infty} p_{i,i}^{(n)} &= 1 \qquad \forall i \in \{r + 1, \dots, N\}\end{aligned}$$

Then we can express the transition matrix P as

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix},$$

where Q and R is some transition matrices for the corresponding partitioned states and 0 is the zero matrix and I is the identity matrix.

Let $T = \min\{n \geq 0 : X_n \geq r+1\}$ be the time of absorption and X_T be the state we get absorbed into. Define $u_{i,k} = \mathbb{P}(X_T = k \mid X_0 = i)$. Then we have

$$\begin{aligned} u_{i,k} &= \sum_{j=1}^{N} \mathbb{P}(X_T = k, X_1 = j \mid X_0 = i) \\ &= \sum_{j=1}^{N} \mathbb{P}(X_T = k \mid X_1 = j, X_0 = i)\mathbb{P}(X_1 = j \mid X_0 = i) \\ &= \sum_{j=1}^{N} p_{i,j}\mathbb{P}(X_T = k \mid X_1 = j) \\ &= \sum_{j=1}^{N} \mathbb{P}(X_T = k \mid X_0 = j) \\ &= \sum_{j=1}^{r} p_{i,j}u_{j,k} + \sum_{j=r+1, j\neq k}^{N} p_{i,j}u_{j,k} + p_{i,k}u_{k,k}. \end{aligned}$$

Thus,

$$u_{i,k} = \sum_{j=1}^{r} P_{i,j}u_{j,k} + p_{i,k}$$

Hence, we have

$$U = QU + R \implies (I - Q)U = R \implies U = (I - Q)^{-1}R,$$

where U contains all the $(u_{i,k})_{i\in\{1,\ldots,r\},k\in\{r+1,\ldots,N\}}$.

3.2.1 The General Absorbing Markov Chain

Let's suppose that associated with each transient state i is a rate $g(i)$ and that we wish to determine the mean total rate that is accumulated up to absorption. Let v_i be this mean total amount, where the subscript i denotes the starting position $X_0 = i$, i.e.,

$$v_i = \mathbb{E}\left[\sum_{n=0}^{T-1} g(X_n) \mid X_0 = i\right]$$

The choice $g = 1$ will give $v_i = \mathbb{E}[T \mid X_0 = i]$. We can also write for $i \in \{1, \ldots, r\}$ that

$$\begin{aligned} v_i &= g(i) + \mathbb{E}\left[\sum_{n=1}^{T-1} g(X_n) \mid X_0 = i\right] \\ &= g(i) + \sum_{j=1}^{N} p_{i,j}v_j \qquad \left(= \sum_{j=1}^{N} p_{i,j}(g(i) + v_i)\right). \end{aligned}$$

Then we can condense this into the following form

$$v = g + Qv \implies v = (I - Q)^{-1}g.$$

where $v = (v_i)_{i\in\{1,\ldots,r\}}$ and $g = (g(i))_{i\in\{1,\ldots,r\}}$.

3.3 Random Walk

$(\xi_n)_{n=1}^{\infty}$ i.i.d and $\mathbb{Z}$-valued. Then

$$X_n = \sum_{i=0}^{n} \xi_i.$$

$$\begin{aligned}\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1) &= \mathbb{P}(\xi_{n+1} = j - i \mid \xi_n = i - i_{n-1}, \ldots, \xi_1 = i_1)\\ &= \mathbb{P}(\xi_{n+1} = j - i)\\ &= \mathbb{P}(\xi_{n+1} = j - i \mid X_n = i).\end{aligned}$$

Example 3.3.1 (Gambler's Ruin). Win 1 dollar with probability p and lose 1 dollar with probability $q = 1 - p$. Stop when we lose all money or make N dollars. We are interested in $u_k = \mathbb{P}(X_T = 0 \mid X_0 = k)$ and $v_k = \mathbb{E}[T \mid X_0 = k]$. Clearly $u_0 = 1, u_N = 0$. For $k = 1, \ldots, N-1$, we have

$$u_k = pu_{k+1} + qu_{k-1} \implies q(u_k - u_{k-1}) = p(u_{k+1} - u_k)$$

Let $\Delta_{k+1} = u_{k+1} - u_k$. Then we have

$$\begin{aligned}q\Delta_k &= p\Delta_{k+1}\\ \Delta_{k+1} &= \frac{q}{p}\Delta_k = \cdots = \left(\frac{q}{p}\right)^k \Delta_1.\end{aligned}$$

$$\sum_{i=1}^{m} \Delta_i = \Delta_1 \sum_{i=1}^{m} \left(\frac{q}{p}\right)^{i-1} = \sum_{i=1}^{n} u_i - u_{i-1} = u_m - u_0 = u_m - 1$$

Thus,

$$u_m = 1 + \Delta_1 \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \frac{q}{p}} \qquad m = 1, \ldots, N$$

When $m = N$,

$$0 = 1 + \Delta_1 \frac{1 - \left(\frac{q}{p}\right)^N}{1 - \frac{q}{p}} \implies \Delta_1 = -\frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N}.$$

Substituting the expression for Δ_1 gives

$$u_m = 1 + \left(-\frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N}\right)\left(\frac{1 - \left(\frac{q}{p}\right)^m}{1 - \frac{q}{p}}\right) = 1 - \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^N} = \frac{\left(\frac{q}{p}\right)^m - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}.$$

Note that $p \neq q$. If $p = q$, then

$$\sum_{i=1}^{m} \Delta_i = \Delta_1 m = u_m - 1 \implies u_m = \frac{N - m}{N}.$$

If we take limit as $N \to \infty$ for $p \leq q$, then

$$\lim_{N \to \infty} u_m = 1,$$

which implies that we will be broke at the end no matter how much money we started with. If $p > q$, then

$$\lim_{N \to \infty} u_m = \left(\frac{q}{p}\right)^m.$$

If m is large, then this quantity becomes small. This implies that if $p > q$ and we started with a lot of money, then the chance of us being broke ultimately becomes smaller.

Now lets compute v_k when $p = q = \frac{1}{2}$. Clearly, $v_0 = 0$ and $v_N = 0$. For $k = 1, \dots, N-1$, we have

$$v_k = 1 + \frac{1}{2}v_{k+1} + \frac{1}{2}v_{k-1}.$$

Let $\Delta_k = v_k - v_{k-1}$. Then we have

$$0 = 1 + \frac{1}{2}(\Delta_{k+1} - \Delta_k).$$

Summing both sides gives

$$\sum_{k=1}^{m} 0 = m + \sum_{k=1}^{m} \frac{1}{2}(\Delta_{k+1} - \Delta_k) \implies \Delta_1 = 2m + \Delta_{m+1} \qquad m = 0, \dots, N-1.$$

Then

$$\begin{aligned} \sum_{m=0}^{k} \Delta_1 &= \sum_{m=0}^{k} (2m + \Delta_{m+1}) \\ (k+1)\Delta_1 = (k+1)v_1 &= \sum_{m=0}^{k} 2m + \sum_{m=0}^{k} \Delta_{m+1} \\ (k+1)v_1 &= k(k+1) + (v_{k+1} - v_0) \implies (k+1)v_1 = k(k+1) + v_{k+1}. \end{aligned}$$

Take $k = N-1$ gives

$$Nv_1 = (N-1)N + 0 \implies v_1 = N-1.$$

Then

$$v_{k+1} = (k+1)(v_1 - k) = (k+1)(N-1-k).$$

Hence,

$$v_k = k(N-k).$$

3.4 Branching Process

$\left(\xi_i^{(n)}\right)_{i=1,n=0}^{\infty,\infty}$ i.i.d. $\mathbb{N}_0$-valued random variables where $\xi_i^{(n)}$ is the number of offspring of ith individual in nth generation. $X_0 = 1$. $\mathbb{E}[\xi_i] = \mu$ and $\text{Var}(\xi_i) = \sigma^2$. The population of at time $n+1$ is

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$$

Our goal is to compute $\mathbb{P}(X_n = 0 \text{ eventually} \mid X_0 = 1)$. But let's first compute $\mathbb{E}[X_{n+1}]$ and $\text{Var}(X_{n+1})$. Recall that

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{N} \xi_i\right] &= \mathbb{E}[N]\mathbb{E}[\xi_i] \\ \text{Var}\left(\sum_{i=1}^{N} \xi_i\right) &= \text{Var}(N)\mathbb{E}[\xi_i]^2 + \text{Var}(\xi_i)\mathbb{E}[N]. \end{aligned}$$

Then we have

$$\mathbb{E}[X_{n+1}] = \mu\mathbb{E}[X_n] = \mu^{n+1}$$
$$\text{Var}(X_{n+1}) = \mu^2\text{Var}(X_n) + \mu^n\sigma^2.$$

$$c_0 = \text{Var}(X_0) = 0$$
$$c_n = \text{Var}(X_n)$$
$$c_{n+1} = \mu^2 c_n + \mu^n\sigma^2.$$

Define the generating function $f(x)$ as

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{n+1}x^{n+1} &= \mu^2 x\sum_{n=0}^{\infty} c_n x^n + \sigma^2 x\sum_{n=0}^{\infty}\mu^n x^n \\ &= \mu^2 x f(x) + \frac{\sigma^2 x}{1-\mu x}. \end{aligned}$$

Then

$$f(x) = \frac{\sigma^2 x}{(1-\mu x)(1-\mu^2 x)} = \sigma^2 x\frac{1}{1-\mu x}\frac{1}{1-\mu^2 x}.$$

Since

$$\sum_{j=1}^{\infty} c_j x^j = \sigma^2 x\sum_{n=0}^{\infty}\mu^n x^n \cdot \sum_{m=0}^{\infty}\mu^{2m}x^m,$$

the coefficient of $x^{j-1} = \sum_{k=0}^{j-1} x^k x^{j-1-k}$ is

$$c_j = \sum_{k=0}^{j-1}\mu^k\mu^{2(j-1-k)}.$$

Thus

$$\text{Var}(X_n) = \sigma^2\mu^{n-1}\cdot\begin{cases} n & \text{if } \mu = 1 \\ \frac{1-\mu^{n-1}}{1-\mu} & \text{if } u \neq 1. \end{cases}$$

Remark. When $\mu = 1$, expectation is constant, variance is growing linearly. When $\mu \neq 1$, expectation is increasing/decreasing geometrically, same with variance.

Now let $T = \min\{n \geq 0 : X_n\}$ be the time the population dies out and let $u_n = \mathbb{P}(T \leq n) = \mathbb{P}(X_n = 0)$. Then $\lim_{n\to\infty} u_n$ is the probability of extinction.

$$u_{n+1} = \sum_{k=0}^{\infty} p_k u_n^k$$

where $p_k = \mathbb{P}(\xi = k)$. We have $u_0 = 0, u_1 = p_0$.

Let $\phi_\xi : [0,1] \to [0,1]$ be the generating function of ξ defined by

$$\phi_\xi(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{\infty} p_k s^k.$$

Then we have

$$u_{n+1} = \phi(u_n) \implies u_\infty = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty}\phi(u_n) \implies u_\infty = \phi(\lim_{n\to\infty} u_n) = \phi(u_\infty).$$

Thus, u_∞ is a fixed point for ϕ.

3.4.1 Generating Functions

Given any $\mathbb{N}_0$-valued random variable ξ with $p_k = \mathbb{P}(\xi = k)$. Then the generating function is given by

$$\phi_\xi(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{\infty} p_k s^k.$$

ϕ_ξ completely recovers the distribution of ξ. We have $\phi_\xi(0) = p_0 \phi_\xi(1) = 1$. We can recover p_k via

$$p_k = \frac{\phi^{(k)}(0)}{k!}.$$

Then

$$\mathbb{E}[X] = \phi'(1) = \sum_{k=1}^{\infty} k p_k.$$

In fact, one can check that

$$\begin{aligned} \phi''(1) &= \mathbb{E}[X(X-1)] \\ \phi^{(k)}(1) &= \mathbb{E}[X(X-1)\cdots(X-k+1)]. \end{aligned}$$

Suppose $\xi_1, \dots, \xi_n$ i.i.d has generating function ϕ. Then $Z = \sum_{i=1}^n \xi_i$ has the following generating function:

$$\phi_Z(s) = \mathbb{E}[s^Z] = \mathbb{E}[s^{\sum_{i=1}^n \xi_i}] = \prod_{i=1}^n \mathbb{E}[s^{\xi_i}] = \phi^n(s).$$

But if instead we have $Z = \sum_{i=1}^N \xi_i$ where N is a random variable and N has gnerating function g_N. Then the generating function would be

$$\begin{aligned} \mathbb{E}[s^{\sum_{i=1}^N \xi_i}] &= \sum_{n=0}^{\infty} \mathbb{P}(N = n)\phi^n(s) \\ &= g_N(\phi(s)). \end{aligned}$$

Now suppose $\phi_n(s)$ is the generating function of X_n defined by

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

Then applying the result from above, we have

$$\phi_{n+1}(s) = \phi_n(\phi(s)) = \phi^{(n+1)}(s).$$

4 The Long Run Behavior of Markov Chains

4.1 Regular Transition Probability Matrices

Suppose $(X_n)_{n=0}^{\infty}$ is a Markov Chain on $\{1, \ldots, N\}$.

Definition 4.1.1 (Regular). $(X_n)_{n=0}^{\infty}$ is *regular* if $\exists m \geq 1$ such that P^m has all positive entries.

Theorem 4.1.2.
If $(X_n)_{n=0}^{\infty}$ is regular, there exists a limiting distribution $\hat{\pi} = (\pi_1, \ldots, \pi_N)$, where $\pi_i > 0$ and $\sum_{i=1}^{N} \pi_i = 1$ such that

$$\lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i) = \pi_j, \qquad \forall i, j \in \{1, \ldots, N\}.$$

This limiting distribution does not depend on initial distribution.

Corollay 4.1.3. Suppose $\mathbb{P}(X_0 = i) = \alpha_i$. Then

$$\lim_{n\to\infty} \mathbb{P}(X_n = j) = \pi_j > 0.$$

Question. How do we find π?

Theorem 4.1.4.
π is the unique solution to $\pi P = \pi$ satisfying $\langle \hat{\pi}, \hat{1} \rangle = \sum_{i=1}^{N} \pi_i = 1$ and $\pi_i \geq 0$ for all i.

Proof. We first check that π is a solution.

$$\begin{aligned} \pi &= \lim_{n\to\infty} \pi P^n \\ \pi P &= \lim_{n\to\infty} \pi P^{n+1} = \lim_{m\to\infty} \pi P^m = \pi. \end{aligned}$$

Now we check for uniqueness. Let τ be any distribution that satisfies $\tau P = \tau$. Then

$$\begin{aligned} \lim_{n\to\infty} \tau P^n &= \pi \\ \lim_{n\to\infty} \tau &= \pi \\ \tau &= \pi. \end{aligned}$$

□

4.2 Doubly Stochastic Matrices

Definition 4.2.1 (Doubly stochastic). A matrix is *doubly stochastic* if every row and column sums to 1.

Proposition 4.2.2. If (X_n) is doubly stochastic, then

$$\pi = \left(\frac{1}{N}, \cdots, \frac{1}{N}\right).$$

Proof.

$$\begin{aligned}
\left(\frac{1}{N}, \cdots, \frac{1}{N}\right) P &= \left(\frac{1}{N}, \cdots, \frac{1}{N}\right) \begin{pmatrix} P_{1,1} & \cdots & \vdots \\ P_{2,1} & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ P_{N,1} & \cdots & \vdots \end{pmatrix} \\
&= \left(\frac{1}{N}\sum_{i=1}^{N} P_{i,1}, \cdots, \frac{1}{N}\sum_{i=1}^{N} P_{i,m}\right) \\
&= \left(\frac{1}{N}, \cdots, \frac{1}{N}\right).
\end{aligned}$$

□

4.3 Interpretation of π

- $\pi_j = \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i) = \lim_{n\to\infty} P^n_{i,j}$.
- π_j is the mean fraction of time the process spends in state j.

$$\begin{aligned}
\pi_j &= \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n+1}\sum_{m=0}^{n} \mathbf{1}\{X_m = j\} \mid X_0 = i\right] \\
&= \lim_{n\to\infty} \frac{1}{n+1}\sum_{m=0}^{n} P^m_{i,j} \\
&= \pi_j.
\end{aligned}$$

4.4 Irreducible Markov Chains

Definition 4.4.1 (Accessible). State j is *accessible* from state i if there exists n such that $P^{(n)}_{i,j} > 0$.

Definition 4.4.2 (Irreducible). If $\forall i, j \in S$, and $i \leftrightarrow j$ (i and j communicate with each other), we say that $(X_n)_{n\geq 0}$ is *irreducible*.

4.4.1 Recurrent and Transient States

Let $f_{i,i}^{(n)}$ be the probability of first return to i at step n given that we started at i at step 0, i.e.,

$$f_{i,i}^{(n)} = \mathbb{P}(X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i \mid X_0 = i).$$

We have $f_{i,i}^{(0)} = 0$.

Claim. For $n \geq 1$,

$$P_{i,i}^{(n)} = \sum_{k=0}^{n} f_{i,i}^{(k)} P_{i,i}^{(n-k)} = \sum_{k=1}^{n} f_{i,i}^{(k)} P_{i,i}^{(n-k)}.$$

Proof. Let E_k be the event that the first return to i is at time k. Then

$$\begin{aligned} P_{i,i}^{(n)} &= \mathbb{P}(X_n = i \mid X_0 = i) \\ &= \sum_{k=1}^{n} \mathbb{P}(X_n = i, E_k \mid X_0 = i) \\ &= \sum_{k=1}^{n} \mathbb{P}(X_n = i \mid E_k, X_0 = i)\mathbb{P}(E_k \mid X_0 = i) \\ &= \sum_{k=1}^{n} \mathbb{P}(X_n = i \mid X_k = i) f_{i,i}^{(k)} \\ &= \sum_{k=1}^{n} P_{i,i}^{(n-k)} f_{i,i}^{(k)}. \end{aligned}$$

□

Question. What is the chance of returning to i eventually?

Answer. $\sum_{n=0}^{\infty} f_{i,i}^{(n)}$.

Definition 4.4.3 (Recurrent)**.** State i is *recurrent* if and only if $f_{i,i} := \sum_{n=0}^{\infty} f_{i,i}^{(n)} = 1$.

Definition 4.4.4 (Transient)**.** State i is *transient* if and only if $f_{i,i} < 1$.

Let $M = \sum_{n=1}^{\infty} \mathbf{1}\{X_n = i\}$ be the number of returns to i. If i is recurrent, then

$$\mathbb{E}[M \mid X_0 = i] = \infty.$$

If i is transient, then

$$\begin{aligned} \mathbb{E}[M \mid X_0 = i] &= \sum_{m=1}^{\infty} \mathbb{P}(M \geq m \mid X_0 = i) \\ &= \sum_{m=1}^{\infty} f_{i,i}^{(m)} \\ &= \frac{f_{i,i}}{1 - f_{i,i}}. \end{aligned}$$

Theorem 4.4.5.
A state i is recurrent if and only if

$$\sum_{n=1}^{\infty} P_{i,i}^{(n)} = \infty.$$

Equivalently, i is transient if and only if

$$\sum_{n=1}^{\infty} P_{i,i}^{(n)} < \infty.$$

Proof. i is transient $\iff \mathbb{E}[M \mid X_0 = i] < \infty \iff \sum_{n=1}^{\infty} P_{i,i}^{(n)} < \infty$. □

Proposition 4.4.6. If $i \leftrightarrow j$, then i recurrent $\iff$ j recurrent.

Proof. We know that $P_{ij}^{(n)} > 0$ and $P_{ji}^{(m)} > 0$. Note that

$$P_{j,j}^{(m+k+n)} \geq P_{j,i}^{(m)} P_{i,i}^{(k)} P_{i,j}^{(n)}$$
$$\sum_k P_{j,j}^{(m+k+n)} \geq \sum_k P_{j,i}^{(m)} P_{i,i}^{(k)} P_{i,j}^{(n)} = P_{j,i}^{(m)} \left(\sum_k P_{i,i}^{(k)} \right) P_{i,j}^{(n)} \geq \infty.$$

□

4.5 Periodicity

Definition 4.5.1 (Period)**.** For $i \in S$,

$$d(i) = gcd\{n : P_{i,i}^{(n)} > 0\}$$

is the *period* of i.

Remark. $d(i) \neq \min_n \{n : P_{i,i}^{(n)} > 0\}$.

Fact.

1. $i \leftrightarrow j \implies d(i) = d(j)$.
2. $\exists N, \forall n \geq N, P_{i,i}^{(nd(i))} > 0$.
3. $P_{j,i}^{(m)} > 0 \implies P_{j,i}^{(m+nd(i))} > 0$ for $n \geq N$.

Definition 4.5.2 (Aperiodic)**.** Assume a MC is irreducible. If $d(i) = 1$ for some $i \in S$, then the MC is *aperiodic*.

Theorem 4.5.3.
$(X_n)_{n=0}^{\infty}$ regular $\iff$ $(X_n)_{n=0}^{\infty}$ irreducible and aperiodic.

Let $R_i = \min\{n \geq 1 : X_n = i\}$. Then

$$\mathbb{P}(R_i = k \mid X_0 = i) = f_{i,i}^{(k)}.$$

If i is recurrent,

$$\mathbb{P}(R_i < \infty) = \sum_k f_{i,i}^{(k)} = 1.$$

Theorem 4.5.4.
Assume (X_n) aperiodic, irreducible, and recurrent, define

$$\mathbb{E}[R_i \mid X_0 = i] = m_i,$$

which is the mean time of first return. Then

$$\lim_{n\to\infty} P_{i,i}^{(n)} = \lim_{n\to\infty} P_{j,i}^{(n)} = \frac{1}{m_i}.$$

Definition 4.5.5 (Positive/null recurrent)**.** If $m_i < \infty$, the MC is *positive recurrent*. Otherwise, it is *null recurrent*.

Proposition 4.5.6.

$$\prod_{i=0}^{\infty}(1 - p_i) = 0 \iff \sum_{i=0}^{\infty} p_i = \infty.$$

Theorem 4.5.7.
If $(X_n)_{n=0}^{\infty}$ is positive recurrent, aperiodic, and irreducible, then π is a limiting distribution that is the unique solution to

$$\pi = \pi P, \qquad \sum_i \pi_i = 1, \quad \pi_i \geq 0.$$

5 Poisson Process

Recall that Poisson counts the number of occurrences of a rare event.

5.1 The Law of Rare Events

Consider

$$\text{Bin}\left(n, \frac{\lambda}{n}\right) \overset{D}{=} X_n.$$

$$\begin{aligned}\mathbb{E}[X_n] &= \lambda \\ \lim_{n\to\infty} \mathbb{P}(X_n = k) &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{e^{-\lambda}\lambda^k}{k!}.\end{aligned}$$

5.2 Poisson Process

Idea: count the number of occurrences up to a certain time.

Definition 5.2.1 (Poisson Process). The $\mathbb{N}_0$-valued process $(N_t)_{t\geq 0}$ is a $PP(\lambda)$ if

(i) $N_0 = 0$,

(ii) Increments are independent: for any $t_0 < t_1 < \cdots < t_n$,

$$N_{t_n} - N_{t_{n-1}}, \cdots, N_{t_1} - N_{t_0}$$

are independent,

(iii) $N_{t+h} - N_t \sim \text{Poisson}(\lambda h)$.

Example 5.2.2. Customers arriving to a store with rate $\lambda = 10$/hour. Store opens at 8am. What is the probability that 4 customers arrived by noon and 10 by 4pm?

$$\mathbb{P}(N_4 = 4, N_8 = 10) = \mathbb{P}(N_8 - N_4 = 6, N_4 = 4) = \mathbb{P}(\text{Poisson}(4\lambda) = 6)\mathbb{P}(\text{Poisson}(4\lambda) = 4)$$

Question. Why the $PP(\lambda)$?

Answer. Strong uniqueness and computationally tractable.

$$\mathbb{P}(N_{t+h} - N_t = 1) = e^{-\lambda h}\lambda h.$$

$$\lim_{h\to\infty} \frac{\mathbb{P}(N_{t+h} - N_t = 1)}{h} = \lim_{h\to\infty} \lambda e^{-\lambda h} = \lambda.$$

$$\begin{aligned}
\frac{\mathbb{P}(N_{t+h} - N_t \geq 2)}{h} &= \lambda e^{-\lambda h} \sum_{k=2}^{\infty} \frac{(\lambda h)^{k-1}}{k!} \\
&= \lambda e^{-\lambda h} \sum_{k=1}^{\infty} \frac{(\lambda h)^{k}}{(k+1)!} \\
&\leq \lambda e^{-\lambda h} \sum_{k=1}^{\infty} \frac{(\lambda h)^{k}}{k!} \\
&= \lambda e^{-\lambda h}(e^{\lambda h} - 1) \to 0.
\end{aligned}$$

Remark. This shows that it is impossible to have more than two arrivals at the exact same time.

Question. What if

$$\lim_{h\to 0} \frac{\mathbb{P}(N_{t+h} - N_t = 1)}{h} = \lambda(t) \neq \lambda?$$

Answer. This can be done by reducing to a time shift of homogenenous Poisson Process.

5.3 Nonhomogeneous Poisson Process

Definition 5.3.1 (Nonhomogeneous Poisson Process)**.** Same assumptions with homogeneous Poisson Process except that we have a rate function $\lambda(t)$ and that

$$N_{t+h} - N_t \sim \text{Poisson}\left(\int_t^{t+h} \lambda(u)du\right).$$

In fact when $\lambda(u)$ is constant, we can recover a homogeneous Poisson Process.

5.3.1 Time change

Suppose we have a continuous Poisson Process $(N_t)_{t\geq 0}$ with $\lambda(t) > 0$. Define

$$\Lambda(t) = \int_0^t \lambda(u)du.$$

Let $Y_s = X_{\Lambda^{-1}(s)}$. Let's check that this PP is homogeneous.

$$\begin{aligned}
Y_{s+h} - Y_s &= X_{\Lambda^{-1}(s+h)} - X_{\Lambda^{-1}(s)} \\
&\stackrel{D}{=} PP\left(\int_{\Lambda^{-1}(s)}^{\Lambda^{-1}(s+h)} \lambda(u)du\right) \\
&= PP\left(\int_0^{\Lambda^{-1}(s+h)} \lambda(u)du - \int_0^{\Lambda^{-1}(s)} \lambda(u)du\right) \\
&= PP\left(\Lambda(\Lambda^{-1}(s+h)) - \Lambda(\Lambda^{-1}(s))\right) \\
&= PP(s+h-s) \\
&= PP(h).
\end{aligned}$$

Theorem 5.3.2.
Let $(N_t)_{t\geq 0}$ $\mathbb{N}_0$-valued be a stochastic process such that

(i) $N_0 = 0$,

(ii) increments are independent,

(iii) $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)$ as $h \downarrow 0$,

(iv) $\mathbb{P}(N_{t+h} - N_t \geq 2) = o(h)$ as $h \downarrow 0$.

Then $(N_t)_{t\geq 0}$ is $PP(\lambda)$.

Lemma 5.3.3. If $\epsilon \sim \text{Ber}(p_i), \mu = \sum_{i=1}^n p_i, S_n = \sum_{i=1}^n \epsilon_i, X_n \sim \text{Poisson}(\mu)$, then

$$|\mathbb{P}(S_n = k) - \mathbb{P}(X_n = k)| \leq \sum_{i=1}^n p_i^2$$

Proof.

$$X_n = \sum_{i=1}^n Y_i \qquad Y_i \sim \text{Poisson}(p_i).$$

Define $C = \{\epsilon_i = Y_i \text{ for all } i\}$. Then

$$\begin{aligned}
|\mathbb{P}(S_n = k, C) - \mathbb{P}(X_n = k, C) + \mathbb{P}(S_n = k, C^c) - \mathbb{P}(X_n = k, C^c)| &= |\mathbb{P}(S_n = k, C^c) - \mathbb{P}(X_n = k, C^c)| \\
&\leq \mathbb{P}(C^c) \\
&\leq \sum_{i=1}^n \mathbb{P}(\epsilon_i \neq Y_i) \\
&\leq \sum_{i=1}^n p_i^2.
\end{aligned}$$

The last line follows because $\mathbb{P}(\epsilon \neq Y) \leq p^2 \implies \mathbb{P}(\epsilon = Y) \geq 1 - p^2$. ∎

5.4 The Law of Rare Events (cont'd)

$$\text{Bin}\left(n, \frac{\lambda}{n}\right) \xrightarrow{D} \text{Poisson}(\lambda) \qquad \text{as } n \to \infty.$$

What about the error?
Consider $\epsilon_i \sim \text{Ber}(p_i)$. Then

$$\mathbb{P}\left(\sum_{i=1}^{n} \epsilon_i = k\right) = \sum_{x_1+\cdots+x_n=k, x_i \in \{0,1\}} \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i}.$$

Theorem 5.4.1.
Suppose $(M_t)_{t\geq 0}$ is a counting process such that

(i) $M_0 = 0$,

(ii) independent increments,

(iii) distribution of $M_s - M_t$ only depends on $s - t$,

(iv) $\mathbb{P}(M_{t+h} - M_t = 1) = \lambda h + o(h)$,

(v) $\mathbb{P}(M_{t+h} - M_t \geq 2) = o(h)$.

Then $(M_t)_{t\geq 0}$ is a $PP(\lambda)$.

Proof. It suffices to show $\mathbb{P}(M_t = k) - \mathbb{P}(\text{Poisson}(\lambda t) = k) = 0$.
Idea:

$$\begin{aligned} M_t &= \sum_{i=1}^{n} M_{ti/n} - M_{t(i-1)/n} \\ &\approx \sum_{i=1}^{n} \mathbf{1}_{M_{ti/n} - M_{t(i-1)/n} = 1} && \text{(by (v))} \\ &\approx \text{Poisson}(\lambda t + o(t)) && \text{(by (iv))} \\ &\to \text{Poisson}(\lambda t). \end{aligned}$$

$$\begin{aligned} \left|\mathbb{P}\left(\sum_{i=1}^{n} M_{ti/n} - M_{t(i-1)/n} = k\right) - \mathbb{P}\left(\sum_{i=1}^{n} \mathbf{1}_{M_{ti/n} - M_{t(i-1)/n} = k}\right)\right| &\leq \sum_{i=1}^{n} \mathbb{P}(M_{ti/n} - M_{t(i-1)/n} \neq \mathbf{1}_{M_{ti/n} - M_{t(i-1)/n} = 1}) \\ &= \sum_{i=1}^{n} o\left(\frac{t}{n}\right) \\ &= o(t) \qquad \text{as } n \to \infty. \end{aligned}$$

□

5.5 Waiting time distribution

Let W_n be the waiting time for the nth arrival. Then

$$\begin{aligned}\mathbb{P}(W_n \geq t) &= \mathbb{P}(N_t \leq n-1) \\ &= \sum_{k=0}^{n-1} \mathbb{P}(N_t = k) \\ &= \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}.\end{aligned}$$

Then taking derivative gives

$$\begin{aligned}-\lambda e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \sum_{k=1}^{n-1} \frac{(\lambda t)^{k-1}}{(k-1)!} &= -\lambda e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \sum_{k=0}^{n-2} \frac{(\lambda t)^k}{k!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t \geq 0,\end{aligned}$$

which is exactly the density of $\text{Gamma}(n, \lambda)$.

Consider $n = 1$. We have $W_1 \sim \text{Exp}(\lambda)$.

Corollay 5.5.1. Let $S_n = W_{n+1} - W_n$ be the nth interarrival time. Then $S_n \sim \text{Exp}(\lambda)$.

Theorem 5.5.2.
Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. $\text{Exp}(\lambda), T_n = \sum_{i=1}^{n} \xi_i$. Define $N_t = \max\{n : T_n \leq t\}$ (the most people you can jam in by time t). Then $(N_t)_{t \geq 0}$ is $PP(\lambda)$.

Proof. We need to show the following:

- ${}_0 = 0$.

  Proof. Trivial. □

- $N_u \sim \text{Poisson}(\lambda u)$.

Proof. $N_h \overset{D}{=} N_{t+h} - N_t \overset{D}{=} \text{Poisson}(\lambda h)$.

$$\begin{aligned}
\mathbb{P}(T_n \le u < T_{n+1}) &= \mathbb{P}(T_n \le u < T_n + \xi_{n+1}) \\
&= \int_0^u \int_{u-T}^\infty \lambda e^{-\lambda\xi} \lambda e^{-\lambda T} \frac{(\lambda T)^{n-1}}{(n-1)!} d\xi dT \\
&= \int_0^u \lambda e^{-\lambda T} \frac{(\lambda T)^{n-1}}{(n-1)!} \int_{u-T}^\infty \lambda e^{-\lambda\xi} d\xi dT \\
&= \int_0^u \lambda e^{-\lambda T} \frac{(\lambda T)^{n-1}}{(n-1)!} e^{-\lambda(u-T)} dT \\
&= \int_0^u \lambda e^{-\lambda u} \frac{(\lambda T)^{n-1}}{(n-1)!} dT \\
&= e^{-\lambda u} \frac{(\lambda u)^n}{n!} \\
&= \mathbb{P}(\text{Poisson}(\lambda u) = n).
\end{aligned}$$

□

- $(N_{t+s} - N_s)_{t\ge 0}$ is independent of $(N_r)_{0\le r\le s}$ and has the same distribution as $(N_t)_{t\ge 0}$.

  Proof.

$$\begin{aligned}
\mathbb{P}(T_{n+1} > w \mid N_u = n) &= \frac{\mathbb{P}(T_{n+1} > w, N_u = n)}{\mathbb{P}(N_u = n)} \\
&= \frac{\mathbb{P}(T_n \le u, w < T_{n+1})}{\mathbb{P}(N_u = n)} \\
&= \frac{\mathbb{P}(T_n \le u, w, T_n + \xi_{n+1})}{\mathbb{P}(N_u = n)} \\
&= \frac{\int_0^u \int_{w-T} \lambda e^{-\lambda x} \lambda e^{-\lambda T} \frac{(\lambda T)^{k-1}}{(k-1)!} dx dT}{e^{-\lambda u} \frac{(\lambda u)^n}{n!}} \\
&= e^{-\lambda(w-u)}.
\end{aligned}$$

□

□

For $u \le t$,

$$\begin{aligned}
\mathbb{P}(N_u = k \mid N_t = n) &= \frac{\mathbb{P}(N_t = n, N_u = k)}{\mathbb{P}(N_t = n)} \\
&= \frac{\mathbb{P}(N_t = n \mid N_u = k)\mathbb{P}(N_u = k)}{\mathbb{P}(N_t = n)} \\
&= \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}.
\end{aligned}$$

When $n = k = 1$,

$$\mathbb{P}(N_u = 1 \mid N_t = 1) = \frac{u}{t}.$$

This implies that the n arrivals are i.i.d. uniform $[0, t]$.

Question. What does it mean for the arrival times to be uniform?

Answer. Suppose $W_1, \ldots, W_n$ are the arrival times. Then they must satisfy $W_1 \leq W_2 \leq \cdots \leq W_n$. Let $U_1, \ldots, U_n$ be i.i.d. uniform on $[0, t]$. Define $V_1, \ldots, V_n$ where V_i is the ith smallest of the U_i.

Theorem 5.5.3.
If $w_1 \leq \cdots \leq w_n$,

$$f_{W_1,\ldots,W_n|N_t}(w_1, \ldots, w_n \mid n) = f_{V_1,\ldots,V_n}(w_1, \ldots, w_n) = \frac{n!}{t^n}.$$

Proof.

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n)$$
$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{X_1,\ldots,X_n} = f_{X_1,\ldots,X_n}.$$

$$\int_{x_1}^{x_1+\Delta x_1} \cdots \int_{x_n}^{x_n+\Delta x_n} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) dx_n \cdots dx_1 = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\Delta x_1 \cdots \Delta x_n + o(\Delta x_1 \cdots \Delta x_n)$$

Lemma 5.5.4.

$$\lim_{\max \Delta x_i \downarrow 0} \frac{\mathbb{P}(X_1 \in (x_1, x_1 + \Delta x_1], \ldots, X_n \in (x_n, x_n + \Delta x_n])}{\Delta x_1 \cdots \Delta x_n} = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n).$$

$$\begin{aligned}\frac{\mathbb{P}(V_1 \in (v_1, v_1 + \Delta v_1], \cdots V_n \in (v_n, v_n + \Delta v_n])}{\Delta v_1 \cdots \Delta v_n} &= \frac{n!\mathbb{P}(U_1 \in (v_1, v_1 + \Delta v_1], \ldots, U_n \in (v_n, v_n + \Delta v_n])}{\Delta v_1 \cdots \Delta v_n} \\ &= \frac{n! \frac{\Delta v_1}{t} \cdots \frac{\Delta v_n}{t}}{\Delta v_1 \cdots \Delta v_n}\end{aligned}$$

Then

$$\lim_{\max \Delta v_i \downarrow 0} \frac{n!}{t^n} = \frac{n!}{t^n}.$$

Now we prove the other equality by considering all the independent increments:

$$\begin{aligned}\frac{\mathbb{P}(W_1 \in (w_1, w_1 + \Delta w_1], \ldots, W_n \in (w_n, w_n + \Delta w_n] \mid N_t = n)}{\Delta w_1 \cdots \Delta w_n \mathbb{P}(N_t = n)} &= \frac{e^{-\lambda t}\lambda^n \Delta w_1 \cdots \Delta w_n}{\Delta w_1 \cdots \Delta w_n e^{-\lambda t}\frac{(\lambda t)^n}{n!}} \\ &= \frac{n!}{t^n}.\end{aligned}$$

□

Example 5.5.5. Monkeys arrive to airport according to $PP(\lambda)$. Assume that if monkeys arrive within 30 minutes of each other, they fight. Assuming $N_1 = 2$, what are the chances of a fight? (t is in hours)

$$\begin{aligned}\mathbb{P}\left(W_2 - W_1 < 0.5 \mid N_1 = 2\right) &= \mathbb{P}(V_2 - V_1 < 0.5) \\ &= \frac{3}{4}.\end{aligned}$$

5.5.1 Symmetric Functions

Definition 5.5.6 (Symmetric functions). A function $f : \mathbb{R}^n \to \mathbb{R}$ is *symmetric* if

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \qquad \forall \sigma \in S_n,$$

i.e. order of input doesn't matter.

Question. Why do we care about symmetric functions?

If $V_1, \ldots, V_n$ are the order statistics, then there is a random permutation:

$$(V_1, \ldots, V_n) = (U_{\sigma(1)}, \ldots, U_{\sigma(n)}).$$

If f is symmetric, then

$$f(V_1, \ldots, V_n) = f(U_{\sigma(1)}, \ldots, U_{\sigma(n)}) = f(U_1, \ldots, U_n).$$

Example 5.5.7. Consider customers arrival $(N_t)_{t\geq 0}$ as $PP(\lambda)$. When customers arrive, pay \$1. We want to evaluate the expected value of the total sum collected during the interval $(0, t]$ discounted back to time 0.

$$\begin{aligned}
M_t &= \mathbb{E}\left[\sum_{i=1}^{N_t} e^{-\beta W_i}\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\sum_{i=1}^{k} e^{-\beta W_i} \mid N_t = k\right] \mathbb{P}(N_t = k) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\sum_{i=1}^{k} e^{-\beta V_i}\right] \mathbb{P}(N_t = k) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\sum_{i=1}^{k} e^{-\beta U_i}\right] \mathbb{P}(N_t = k) \quad \text{(symmetric function)} \\
&= \left(\sum_{k=0}^{\infty} k\mathbb{P}(N_t = k)\right) \mathbb{E}[e^{-\beta U_1}] \\
&= \lambda t \int_0^t \frac{1}{t} e^{-\beta u} du \\
&= \lambda t \cdot \frac{1 - e^{-\beta t}}{\beta t} \\
&= \frac{\lambda}{\beta}(1 - e^{-\beta t}).
\end{aligned}$$

Example 5.5.8. Given $(N_t)_{t\geq 0}$. Suppose M_t is the number of customers that are still in the store at time t. Once kth customer arrives, stay Y_k amount of time where Y_k are i.i.d. with CDF G. What is M_t in terms of N_t and $(Y_i)_{i=1}^{\infty}$? What is the distribution of M_t?

$$M_t = \sum_{i=1}^{N_t} \mathbf{1}\{W_i + Y_i \geq t\}$$

$$
\begin{aligned}
\mathbb{P}(M_t = m) &= \sum_{n=0}^{\infty} \mathbb{P}(M_t = m \mid N_t = n)\mathbb{P}(N_t = n) \\
&= \sum_{n=m}^{\infty} \mathbb{P}(M_t = m \mid N_t = n)\mathbb{P}(N_t = n) \\
&= \sum_{n=m}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} \mathbf{1}\{W_i + Y_i > t\} = m \mid N_t = n\right)\mathbb{P}(N_t = n) \\
&= \sum_{n=m}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} \mathbf{1}\{V_i + Y_i > t\} = m\right) e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n=m}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} \mathbf{1}\{V_i > t - Y_i\} = m\right) e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n=m}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} \mathbf{1}\{U_i > t - Y_i\} = m\right) e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n=m}^{\infty} \mathbb{P}\left(\mathrm{Bin}(n,p) = m\right) e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \frac{e^{-\lambda t}}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} p^m (1-p)^{n-m} (\lambda t)^{n-m} \frac{(\lambda t)^m}{n!} \\
&= \frac{e^{-\lambda t}}{m!} (\lambda p t)^m \sum_{n=m}^{\infty} \frac{(1-p)^{n-m}(\lambda t)^{n-m}}{(n-m)!} \\
&= \frac{e^{-\lambda t}}{m!} (\lambda p t)^m e^{(1-p)\lambda t} \\
&= e^{-\lambda p t}\frac{(\lambda p t)^m}{m!} \\
&= \mathbb{P}(\mathrm{Poisson}(\lambda p t) = m).
\end{aligned}
$$

Hence, $M_t \sim \mathrm{Poisson}(\lambda p t)$ and $N_t \sim \mathrm{Poisson}(\lambda t)$.
Note that $p = \mathbb{P}(U_i > t - Y_i)$ and

$$
\begin{aligned}
\mathbb{P}(U_i > t - Y_i) &= \frac{1}{t}\int_0^t \mathbb{P}(u + Y_i > t) du \\
&= \frac{1}{t}\int_0^t 1 - \mathbb{P}(Y_i \leq t - u) du \\
&= \frac{1}{t}\int_0^t 1 - G(t-u) du \\
&= \frac{1}{t}\int_0^t 1 - G(u) du.
\end{aligned}
$$

5.6 Thinning

Fact. $N \sim \mathrm{Poisson}(\lambda), X \sim \mathrm{Bin}(N,p) \implies X \sim \mathrm{Poisson}(\lambda p)$.

Fact. $(N_t)_{t\geq 0} \sim PP(\lambda), X_t \sim \mathrm{Bin}(N_t, p) \implies (X_t)_{t\geq 0} \sim PP(\lambda p)$.

Example 5.6.1. Every customer makes a choice $(Y_i)_{i=1}^{\infty}$ i.i.d. where $Y_i \in \{1, \dots, m\}$. Let $(N_j(t))_{t \geq 0}$ be the number of customers that arrived by time t and picked j, i.e.,

$$N_j(t) = |\{i \leq N(t) : Y_i = j\}|.$$

Then we have

$$\sum_{j=1}^{m} N_j(t) = N(t).$$

Here we have

1. $(N_j(t))_{t \geq 0} \sim PP(\lambda \mathbb{P}(Y = j)) = PP(\lambda p_j)$.
2. $((N_j(t))_{t \geq 0})_{j=1}^{m}$ are independent processes.

Let's check these statements by showing the following:

1. $N_j(0) = 0$.

   Proof. $N_j(t) \leq N(t)$. $N_j(0) \leq N(0) = 0$. □

2. N_j has independent increments.
3. $N_j(t+h) - N_j(t) \sim \text{Poisson}(\lambda h p_j)$
4. $(N_j)_{j=1}^{m}$ are independent.

   Proof. Suppose we have $(N_1(t+h) - N_1(t), N_2(t+h) - N_2(t)) = (a, b)$. Then

   $$N(t+h) + N(t) = a + b$$

   $$\begin{aligned} \mathbb{P}((N_1(t+h) - N_1(t), N_2(t+h) - N_2(t)) = (a, b)) &= e^{-\lambda h} \frac{(\lambda h)^{a+b}}{(a+b)!} \binom{a+b}{a} p_1^a p_2^b \\ &= \mathbb{P}(\text{Poisson}(\lambda h p_1) = a) \mathbb{P}(\text{Poisson}(\lambda h p_2) = b). \end{aligned}$$

   □

Theorem 5.6.2.
Assume that an arrival at time s is counted with probability $p(s)$. $(M_t)_{t \geq 0} \sim PP(\lambda p(s))$.

Example 5.6.3. Suppose people arrive to a puzzle solving party according to $(N_t)_{t \geq 0} \sim PP(2)$. The time to solve a puzzle is $U(0, 10)$ i.i.d.. What is the long term distribution of the number of people working on a puzzle? What is the long term probability that there is exactly 1 person who has been working more than 6 minutes and 2 working less than 2 minutes?

Answer.

(a) Recall

$$\lim_{t \to \infty} \mathbb{P}(M_t = n) = \mathbb{P}(\text{Poisson}(\lambda \mathbb{E}[Y]) = n).$$

Therefore, the answer is $\text{Poisson}(2 \cdot 5) = \text{Poisson}(10)$.

(b) $\mathbb{P}(\text{Poisson}(2 \cdot \frac{4}{10}) = 1) \cdot \mathbb{P}(\text{Poisson}(2 \cdot \frac{2}{10}) = 2)$.

5.7 Superposition

Theorem 5.7.1 (Superposition).
Let $(N_k(t))_{k=1}^n$ be independent $(PP(\lambda_k))_{k=1}^n$. Then

$$N(t) = \sum_{k=1}^{n} N_k(t) \sim PP\left(\sum_{k=1}^{n} \lambda_k\right).$$

Proof.

1. $N(0) = \sum_{k=1}^{n} N_k(0) = 0$.

2. Fix $s, t \geq 0$, then

$$\begin{aligned} N(s+t) - N(t) &= \sum_{k=1}^{n} N_k(s+t) - N_k(t) \\ &\sim \sum_{k=1}^{n} \text{Poisson}(\lambda_k s) \\ &= \text{Poisson}\left(s \sum_{k=1}^{n} \lambda_k\right). \end{aligned}$$

3. Check independence between each intervals.

□

Example 5.7.2. Red $PP(\lambda)$ and green $PP(\mu)$. What is the probability to observe 6 red signals before the 4th green signal?

Answer. We merge the two PP to obtain $PP(\lambda + \mu)$. Then the probability of a red showing up is $\lambda/(\lambda + \mu) = p$ and green with $\mu/(\lambda + \mu) = 1 - p$. Then

$$\mathbb{P}(\text{Bin}(9, p) \geq 6) = \sum_{k=6}^{9} \binom{9}{k} p^k (1-p)^k.$$

6 Renewal Process

Definition 6.0.1 (Renewal process)**.** Let F be a CDF such that $F(0) = 0$. Let $(X_i)_i$ be i.i.d random variables with CDF F. Define $W_n = \sum_{i=1}^n X_i$. Then

$$N(t) = \max \left\{ n \mid \sum_{i=1}^{n} X_i \leq t \right\}$$

is a *renewal process*.

Remark. W_n is the nth waiting time. X_n is the nth interrenewal time.

Remark. $(N(s) : s \geq 0)$ is characterized by F. For $0 \leq a < b$,

$$N([a, b]) = |\{k : U_k \in [a, b]\}|$$

$N(t) \geq k \iff W_k \leq t$.

Definition 6.0.2 (Renewal function)**.** The *renewal function* is defined as

$$M(t) = \mathbb{E}[N(t)].$$

Proposition 6.0.3.

$$M(t) = \sum_{k=1}^{\infty} F_k(t) = \sum_{k=1}^{\infty} F^{*n}(t).$$

$$\begin{aligned}
\mathbb{E}[N(t)] &= \sum_{k=1}^{\infty} \mathbb{P}(N(t) \geq k) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(W_k \leq t) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(W_{k-1} + X_k \leq t) \\
&= \sum_{k=1}^{\infty} (F^(n-1) * F)(t) \\
&= \sum_{k=1}^{\infty} F^{*n}(t).
\end{aligned}$$

Theorem 6.0.4.

$$\lim_{t\to\infty} \frac{N(t)}{t} \overset{a.s.}{=} \frac{1}{\mathbb{E}[t_i]},$$

i.e.,

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{N(t)}{t}\right) = 1.$$

Proof. Need SLLN if $(X_i)_{i=1}^{\infty}$ i.i.d. with $\mathbb{E}[X_i] = \mu < \infty$, then

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} \overset{a.s.}{=} \mu.$$

$$\frac{W_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{W_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)}.$$

$$\lim_{t\to\infty} N(t) \overset{a.s.}{=} \infty.$$

Then combine the two to obtain the desired result. □

Theorem 6.0.5.
Assume that each renewal comes with some reward r_i and $(r_i, t_i)_{i=1}^{\infty}$ i.i.d.. Let $R(t)$ be the sum of the rewards accumulated by time t, i.e.,

$$R(t) = \sum_{i=1}^{N(t)} r_i.$$

Then

$$\lim_{t\to\infty} \frac{R(t)}{t} \overset{a.s.}{=} \frac{\mathbb{E}[r_i]}{\mathbb{E}[t_i]}.$$

Proof.

$$\begin{aligned}
\frac{R(t)}{t} &= \frac{R(t)}{N(t)} \frac{N(t)}{t} \\
\lim_{t\to\infty} \frac{R(t)}{t} &= \lim_{t\to\infty} \frac{R(t)}{N(t)} \frac{N(t)}{t} \\
&= \mathbb{E}[r_i] \cdot \frac{1}{\mathbb{E}[t_i]}.
\end{aligned}$$

□

Example 6.0.6. Suppose cars last according to density $h(t)$, L_i. You buy a car if it breaks down or reaches some age T. The cost of a new car is A. The cost of a breakdown is an additional B. What is the long run cost per unit of time of this policy?

Answer. Here $r_i = A + B\mathbf{1}\{L_i < T\}, \quad t_i = \min(L_i, T)$.

$$\begin{aligned}\lim_{t\to\infty}\frac{R(t)}{t} &= \frac{\mathbb{E}[r_i]}{\mathbb{E}[t_i]} \\ &= \frac{A + B\mathbb{P}(L_i < T)}{\int_0^\infty \min(t,T)h(t)dt} \\ &= \frac{A + B\mathbb{P}(L_i < T)}{\int_0^T th(t)dt + \int_T^\infty Th(t)dt} \\ &= \frac{A + B\mathbb{P}(L_i < T)}{\int_0^T th(t)dt + T\mathbb{P}(L_i > T)} \\ &= f(T).\end{aligned}$$

6.1 Alternating renewal process

$(s_i)_{i=1}^\infty$ is lifetime of component with mean μ. $(T_i)_{i=1}^\infty$ is service time to fix a broken component with mean ν.

Question. What proportion of time is the system operational in the long term?

Answer. Apply reward renewal theorem where $r_i = s_i, \quad t_i = s_i + T_i$. Then

$$\lim_{t\to\infty}\frac{R(t)}{t} = \frac{\mathbb{E}[r_i]}{\mathbb{E}[t_i]} = \frac{\mu}{\mu+\nu}.$$

Example 6.1.1. Light bulbs are i.i.d. with μ. We check the closet according to $PP(\lambda)$.

(a) How often are light bulbs changed (in the long term)?

(b) What proportion of time is the light on?

(c) What proportion of visits result in a changed light?

Answer.

(a) $\frac{1}{\mathbb{E}[t_i]} = \frac{1}{\mu+\frac{1}{\lambda}}$.

(b) $\frac{\mathbb{E}[r_i]}{\mathbb{E}[t_i]} = \frac{\mu}{\mu+\frac{1}{\lambda}}$.

(c)

$$\begin{aligned}\lim_{t\to\infty}\frac{R(t)}{N(t)} &= \lim_{t\to\infty}\frac{R(t)}{t}\cdot\frac{t}{N(t)} \\ &= \frac{1}{\mu+\frac{1}{\lambda}}\cdot\frac{1}{\lambda} \\ &= \frac{\frac{1}{\lambda}}{\mu+\frac{1}{\lambda}}.\end{aligned}$$

Example 6.1.2 (Peter principle). If a person is competent at the job gets promoted, otherwise stays at job. So a given job is more likely to be staffed by someone who is incompetent at it. A person is selected uniformly at random for a job, p competent and $1-p$ incompetent. On average, for competent people, they spend μ at this job and for incompetent people, they spend $\nu > \mu$ at the job. What fraction of the time is the job occupied by someone competent?

Answer. Let X_i/Y_i be the time spent by the ith competent/incompetent person on the job if hired. Suppose $B_i \sim$Ber(p) so that

$$B_i = \begin{cases} 1 & i\text{th person is competent} \\ 0 & i\text{th person is incompetent.} \end{cases}$$

Then here we have

$$\begin{aligned} r_i &= B_i X_i \\ t_i &= B_i X_i + (1-B_i)Y_i \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[r_i] &= \mathbb{E}[B_i]\mathbb{E}[X_i] = p\mu \\ \mathbb{E}[t_i] &= \mathbb{E}[B_i]\mathbb{E}[X_i] + \mathbb{E}[1-B_i]\mathbb{E}[Y_i] = p\mu + (1-p)\nu. \end{aligned}$$

Then we conclude the fraction of time the job is occupied by someone competent is

$$\frac{p\mu}{p\mu + (1-p)\nu}.$$

6.2 Queuing theory

6.2.1 GI/G/1

$GI/G/1$ where GI is the general input (t_i is time of arrival), G is the general service time (s_i), and 1 is the number of person handled by service.

Theorem 6.2.1.
Assume t_i i.i.d. with cdf F, and mean $\frac{1}{\lambda}$ and s_i i.i.d. with cdf G and mean $\frac{1}{\mu}$. Then the rate of arrival is λ and the rate of service is μ. Assume $\lambda < \mu$ and that initially there are k customers already in queue, require service times $s_{-1}, s_{-2}, \dots, s_{-k}$ with probability 1, the queue will clear and the fraction of time the worker is busy is $\frac{\lambda}{\mu} < 1$.

Proof. Let $T_n = t_1 + \cdots + t_n$. Let W_{T_n} be the amount of time spend working by time T_n. Then

$$W_{T_n} \le Z_0 + \sum_{i=1}^{n} s_i,$$

where $Z_0 = \sum_{i=-1}^{-k} s_i$. Then

$$\frac{W_{T_n}}{T_n} \le \frac{Z_0 + \sum_{i=1}^{n} s_i}{T_n} = \frac{Z_0 + \sum_{i=1}^{n} s_i}{n} \cdot \frac{n}{T_n} \to \frac{1}{\mu} \cdot \lambda.$$

Taking the limit gives

$$\lim_{t\to\infty} \frac{W_{T_n}}{T_n} \overset{a.s.}{\leq} \frac{\lambda}{\mu}.$$

Since the fraction of time the worker is busy is < 1, the queue must be clear at some point. □

6.2.2 M/G/1

The only difference from $GI/G/1$ is that here we have M being Markovian $PP(\lambda)$. Let X_n be the number of customers in queue when the nth customer is being dealt with service. Suppose $X_1 = k$. Arrivals according to $PP(\lambda)$ and service times have cdf G and with mean $\frac{1}{\mu}$. Then we have the following set up:

$$X_{n+1} = \max(X_n - 1 + \xi_n, 0),$$

where ξ_n is the number of arrivals to the queue when the $n+1$th customer is being dealt with. Now we are interested in the distribution of the number of people arrive ξ_n during a service time $s)n$

$$\begin{aligned}\mathbb{P}(\xi_n = k) &= \int_0^\infty \mathbb{P}(k \text{ arrivals in time } s_n \mid s_n = t) dG(t) \\ &= \int_0^\infty \mathbb{P}(k \text{ arrivals in time } t) dG(t) \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} dG(t).\end{aligned}$$

Let $a_k = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} dG(t)$. Then

$$\mathbb{E}[\xi_n] = \sum_{k=0}^\infty k a_k = \frac{\lambda}{\mu}.$$

7 Continuous-time Markov Chains

Definition 7.0.1 (CTMC). $(X_t)_{t\geq 0}$ is a stationary *CTMC* if for any times $0 = s_0 < s_1 < \cdots < s_m < s, t > 0$

$$\begin{aligned}\mathbb{P}(X_{s+t} = j \mid X_s = i, X_{s_m} = i_m, \cdots, X_{s_0} = i_0) &= \mathbb{P}(X_{s+t} = j \mid X_s = i) \\ &= \mathbb{P}(X_t = j \mid X_0 = i) \\ &= p_t(i,j).\end{aligned}$$

Example 7.0.2 (Poisson Process). $(N_t)_{t\geq 0} PP(\lambda)$ is a CTMC. Take $(Y_n)_{n=0}^{\infty}$ a MC, independent of $(N_t)_{t\geq 0}$. Then $X_t = Y_{N(t)}$. $Y_{N(t)}$ takes length of $N(t)$.

Definition 7.0.3 (Rate). For $i \neq j$, the *rate* is defined as

$$q_{i,j} = \lim_{\epsilon \to 0} \frac{p_\epsilon(i,j)}{\epsilon}.$$

Then the rate matrix is

$$Q(i,j) = \begin{cases} q_{i,j} & i \neq j \\ -\sum_{j\in S} q_{i,j} & i = j. \end{cases}$$

Example 7.0.4. Consider the same example where $X_t = Y_{N(t)}$. Then

$$\begin{aligned}p_t(i,j) &= \sum_{k=0}^{\infty} \mathbb{P}(X_t = j, N_t = k \mid X_0 = i) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(X_t = j \mid N_t = k, X_0 = i)\mathbb{P}(N_t = k \mid X_0 = i) \\ &= \sum_{k=0}^{\infty} u^k(i,j) e^{-\lambda t} \frac{(\lambda t)^k}{k!}.\end{aligned}$$

Then for $i \neq j$,

$$\begin{aligned}\lim_{h\to 0} \frac{p_h(i,j)}{h} &= \lim_{h\to 0} \frac{\sum_{k=1}^{\infty} \mathbb{P}(N_h = k)\mathbb{P}(Y_k = j \mid Y_0 = i)}{h} \\ &= \lambda u_1(i,j).\end{aligned}$$

Example 7.0.5. $PP(\lambda)$. For $n \neq m$,

$$q(n,m) = \lambda \mathbf{1}\{m = n+1\}.$$