Math 185 Notes
Complex Analysis

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Chapter 1

Complex Numbers

1.1 Intro

Suppose we have the a Taylor series as follows:

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}$$

which happens to converge absolutely for $|x| < r$, where r is the *radius of convergence*, i.e.

$$\sum_{n=0}^{\infty} |a_n||x|^n < \infty \quad \text{when } |x| < r, \quad \text{i.e. } x \in (-r, r).$$

Example 1.1.1. The series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges absolutely for $|x| < 1$.

Question. Now what if we replace the real variable x by the complex variable z?

Answer. If $|z| < r$, then

$$\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

so the sum converges absolutely for $z \in D(0, r)$ (disc of radius r centered at zero). This gives a wider range than the real case.

So in this situation, our real-valued series can be extended to a complex-valued series.

Example 1.1.2. Let

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

When viewed as a function $\mathbb{R} \to \mathbb{R}$, $f(z)$ is infinitely differentiable at $z = 0$, and all derivatives of $f(z)$ are zero at $z = 0$. Hence, the Taylor series is

$$f(0) + f'(0) \cdot x + \frac{f"(0)}{2} \cdot x^2 + \cdots = 0 + 0 + 0 + \cdots = 0.$$

So the Taylor series converges to a function different from $f(z)$!

Example 1.1.3. Consider the same example as above, but with z as a complex number. Let $z = it$ where $t \in \mathbb{R}$. Then

$$e^{-1/z^2} = e^{1/t^2},$$

and so

$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0, \\ 0 & t = 0 \end{cases}$$

which is not continuous at $z = 0$ and thus not complex-differentiable at $z = 0$.

Example 1.1.4. Now let's set $z = x + iy$ where $x, y \in \mathbb{R}$. Consider a suitable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which we may view as a function $\mathbb{R}^2 \to \mathbb{R}^2$ (instead of $\mathbb{C} \to \mathbb{C}$). Let's differentiate with respect to x:

$$\begin{aligned} \frac{\partial f(z)}{\partial x} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z) \\ \frac{\partial^2 f(z)}{\partial x^2} &= \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z). \end{aligned}$$

Now with respect to y:

$$\begin{aligned} \frac{\partial f(z)}{\partial y} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = if'(z) \\ \frac{\partial^2 f(z)}{\partial y^2} &= i\frac{\partial f'(z)}{\partial y} = i\frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = -f''(z). \end{aligned}$$

We observe that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) = 0,$$

which means (the real and imaginary parts of) $f(z)$ satisfy the two-dimensional *Laplace equation*.

Thus, complex analysis is a very powerful tool for solving the 2D Laplace equation.

Example 1.1.5. Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \arctan(\infty) - \arctan(-\infty) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

But what about

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx, \quad a \in \mathbb{R}$$

It turns out that this would be quite tricky to compute this with real-valued techniques. But it would be easier using complex techniques (contour integration). Thus, complex analysis is also a powerful too for computing integrals.

Chapter 2

Complex Differentiation

2.1 Derivatives

Definition 2.1.1 (Derivative)**.** The *derivative* of a complex-valued function is

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case.

Recall that for real valued $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \to a} f(x) = L$$

means for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - a| < \delta$. (For any "tolerance" ϵ, we can guarantee $f(x)$ is within ϵ of L by forcing x to be close enough to a.)

Remark. Note that $x = a$ doesn't satisfy $0 < |x - a|$, so the value of f at $x = a$ has no bearing on whether $\lim_{x \to a} f(x)$ exists.

2.1.1 Continuity

Definition 2.1.2 (Continuous)**.** If $\lim_{x \to a} f(x) = f(a)$, then we say f is *continuous* at a.

Remark. Setting $L = f(a)$ in the limit, $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ (even when $x = a$) when talking about continuity, we leave out the $0 < |x-a|$ part for convenience because $x - a = 0$ automatically works.

Now let's consider a function $f : \mathbb{C} \to \mathbb{C}$, $\lim_{z\to a} f(z) = L$ means for every $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |z - a| < \delta \implies |f(z) - L| < \epsilon.$$

Remark. Now the z's that we worry about form an open disc with radius δ instead of an interval from the real case.

Similarly, if $\lim_{z\to a} f(z) = f(a)$, we say f is *continuous* at $z = a$.

Example 2.1.3. $f(z) = z$ is continuous at any point $a \in \mathbb{C}$.

Proof. For $\epsilon > 0$, let $\delta = \epsilon$, then

$$|z - a| < \delta = \epsilon \implies |f(z) - f(a)| < \epsilon.$$

□

Example 2.1.4. $\lim_{z\to 0} \overline{z}/z$ (although this is undefined at $z = 0$, this has no bearing on whether the limit exists).

Proof. Suppose $\lim_{z\to 0} \overline{z}/z = L$ for some L. Let's take $\epsilon = 1$. There is a $\delta > 0$ such that

$$0 < |z - 0| < \delta \implies \left|\frac{\overline{z}}{z} - L\right| < \epsilon = 1.$$

Let $z = \delta/2$ and so does $z = i\delta/2$. Then for $z = \delta/2$:

$$\frac{\overline{z}}{z} = \frac{\delta/2}{\delta/2} = 1 \implies |1 - L| < 1,$$

and for $z = i\delta/2$:

$$\frac{\overline{z}}{z} = \frac{-i\delta/2}{i\delta/2} = -1 \implies |-1 - L| < 1.$$

Thus, we see that the L must lie in the intersection of the two open unit discs centered at -1 and 1. However, since they are open discs, these two discs do not overlap and so L does not exist. □

Remark. This implies that there is no way to extend $\overline{z}/z$ to a continuous function at $z = 0$.

2.1.2 Properties of Limits

If $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$, then

(i)
$$\lim_{x\to a}(f(x) + g(x)) = L_1 + L_2.$$

(ii)
$$\lim_{x\to a} f(x)g(x) = L_1 L_2.$$

(iii)
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

Remark. These implies that the sum/product/quotient of continuous functions are continuous.

Proposition 2.1.5 (Composite function of continuous functions is continuous)**.** *If $f(x)$ is continuous at $x = a$, and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$.*

Proof. We want $|g(f(x)) - g(f(a))| < \epsilon$. By continuity of g at $x = f(a)$, there exists $\delta_1 > 0$ such that
$$|w - f(a)| < \delta_1 \implies |g(w) - g(f(a))| < \epsilon.$$
We want to take $w = f(x)$, so we need
$$|f(x) - f(a)| < \delta_1.$$
But by continuity of f at $x = a$, we know that δ_1 will be our ϵ when $|x - a| < \delta_2$ for some $\delta_2 > 0$. Then for such x,
$$|g(f(x)) - g(f(a))| < \epsilon.$$
□

2.2 Derivatives (Cont'd)

Definition 2.2.1 (Differentiable)**.** We say that $f(z)$ is differentiable at $z = a$ iff $\frac{f(z)-f(a)}{z-a}$ extends to a continuous function at $z = a$ (the value there is $f'(a)$).

Example 2.2.2. $f(z) = z$ is differentiable with $f'(z) = 1$.

Proof.

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{z+h-z}{h}=\lim_{h\to 0}1=1.$$

□

Example 2.2.3 (Interesting one)**.** $f(z)=\overline{z}$ is not differentiable but is continuous.

Proof.

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}&=\lim_{h\to 0}\frac{\overline{z+h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{z}+\overline{h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{h}}{h}\\&=\text{DNE}\qquad\text{(proved in previous example)}\end{aligned}$$

□

Proposition 2.2.4 (Differentiability implies continuity)**.** *$f(z)$ differentiable at $z=a$ implies that $f(z)$ is continuous at $z=a$.*

Proof. We want to show that $\lim_{z\to a}f(z)=f(a)$.

$$\begin{aligned}\lim_{z\to a}f(z)-f(a)&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot(z-a)\\&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot\lim_{z\to a}(z-a)\qquad\text{(assume both limits exist)}\\&=f'(a)\cdot 0\\&=0.\end{aligned}$$

□

Remark. This is a common technique to show continuity by showing the limit of the difference is zero.

2.2.1 Properties of complex-derivatives

(i)
$$\frac{d}{dz}cf(z)=cf'(z),\qquad\forall c\in\mathbb{C}.$$

(ii)
$$\frac{d}{dz}(f+g) = f'(z) + g'(z).$$

(iii)
$$\frac{d}{dz}(fg) = f'g + fg'.$$

(iv)
$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.$$

(v)
$$\frac{d}{dz}f(g(z)) = g'(z)f'(g(z)).$$

Proposition 2.2.5 (Power rule)**.**

$$\frac{d}{dz}z^n = nz^{n-1}$$

for all integers n*.*

Proof. We induct on n. For $n \geq 0$, when $n = 0$,

$$\frac{d}{dz}z^0 = \frac{d}{dz}1 = 0 = 0z^{-1}.$$

By the product rule,

$$\begin{aligned}
\frac{d}{dz}z^n &= \frac{d}{dz}(z \cdot z^{n-1}) \\
&= 1 \cdot z^{n-1} + z \cdot (n-1)z^{n-2} \qquad \text{(inductive hypothesis)} \\
&= nz^{n-1}.
\end{aligned}$$

For $n < 0$, simply apply quotient rule. □

Chapter 3

Holomorphic Functions and Cauchy-Riemann Equations

Recall
$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$
Being differentiable at a point says little about how "nice" a function is.

Example 3.0.1. Consider $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

This is nowhere continuous, and thus nowhere differentiable. But if we consider $x^2 f(x)$, it is differentiable at $x = 0$:
$$\lim_{h\to 0} \frac{h^2 f(h) - 0^2 f(0)}{h} = \lim_{h\to 0} h f(h) = 0.$$
Nevertheless, it's still not a very "nice" function.

3.1 Holomorphic Functions

Definition 3.1.1 (Holomorphic)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* at a point a if it is differentiable at z for all z within distance r of a for some $r > 0$. In other words, $f(z)$ is differentiable everywhere sufficiently close to a.

Definition 3.1.2 (Open/closed disk)**.** The *open disk* of radius r centered at $a \in \mathbb{C}$ is
$$D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$
The *closed disk* is
$$\overline{D}(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Thus, we can say $f(z)$ is holomorphic at $a \in \mathbb{C}$ if $f(z)$ is differentiable on an open disk centered at a. (if the point is not specified, it means that f is holomorphic everywhere.)

Example 3.1.3 (Polynomials are holomorphic)**.** We saw last time that z^n is differentiable everywhere for $n \geq 0$. Then the linear combinations

$$a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

which is a polynomial, is differentiable (since multiplying by constants and summing preserves differentiability).

Example 3.1.4. $f(z) = |z|^2 = z\overline{z}$ is differentiable at zero.

$$\begin{aligned} \lim_{h \to 0} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0} \frac{h\overline{h} - 0}{h} \\ &= \lim_{h \to 0} \overline{h} \\ &= 0. \end{aligned}$$

However, this is not differentiable elsewhere (exercise). Thus, f is not holomorphic.

3.2 The Cauchy-Riemann Equations

Question. How to tell if a function is complex-differentiable?

Answer. We'll reduce this to a question about real derivatives.

Let $x + iy$, where $x, y \in \mathbb{R}$. If $f : \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned} \frac{\partial f}{\partial x}(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h+iy) - f(x+iy)}{h}. \end{aligned}$$

Note that h is real. Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y}(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \to 0} \frac{f(x+i(y+h)) - f(x+iy)}{h}. \end{aligned}$$

Example 3.2.1. $f(z) = z^2$. Then

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

$$\begin{aligned}\frac{\partial f}{\partial x}(z) &= \lim_{h\to 0}\frac{(x+h)^2-y^2+2i(x+h)y-(x^2-y^2+2ixy)}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2+2ihy}{h}\\ &= \lim_{h\to 0} 2x+h+2iy\\ &= 2x+2iy\\ &= 2z\\ &= f'(z).\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}(z) &= \lim_{h\to 0}\frac{x^2-(y+h)^2+2ix(y+h)-(x^2-y^2+2ixy)}{h}\\ &= \lim_{h\to 0}\frac{-2yh-h^2+2ixh}{h}\\ &= \lim_{h\to 0} -2y-h+2ix\\ &= -2y+2ix\\ &= 2i(x+iy)\\ &= if'(z).\end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y}.$$

Theorem 3.2.2.

(i) If $f : \mathbb{C} \to \mathbb{C}$ is complex-differentiable, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and they satisfy

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

(ii) If $f : \mathbb{C} \to \mathbb{C}$ is a function and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous on some open disk centered at z, and if

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y},$$

then f is complex-differentiable at z.

Proof.

(i) Since f is complex-differentiable, we have

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = f'(z).$$

This is equivalent to the statement that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|h - 0| < \delta \implies \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon.$$

Suppose h is real. Then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

and since h is real, we get $\frac{\partial f}{\partial x}$ and thus

$$\frac{\partial f}{\partial x}(z) = f'(z).$$

Now suppose h is purely imaginary: $h = ik$ for $k \in \mathbb{R}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+iy+ik) - f(x+iy)}{ik}.$$

Then $h \to 0$ is equivalent to $k \to 0$ since $|h| = |k|$. Thus we have

$$\lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y} = f'(z).$$

Hence, we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

□

Let $f(z) = u(z) + iv(z)$. If we choose real values for h, then the imaginary part y is kept constant, and the derivative becomes a partial derivative with respect to x. Thus we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Similarly, if we substitute purely imaginary values ik for h, we obtain

$$f'(z) = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Since we have

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

this resolves into the following equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

or simply

$$u_x = v_y, \qquad v_x = -u_y.$$

These are known as the *Cauchy-Riemann* equations.

Example 3.2.3. Consider $f(z) = z^2$. Then

$$f(x+iy) = x^2 + y^2 + 2ixy.$$

Here $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We have

$$u_x = 2x = v_y \qquad v_x = 2y = -u_y.$$

Example 3.2.4. Consider $f(z) = |z|^2$. Then $f(x+iy) = x^2 + y^2$ where $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. But here we have

$$u_x = 2x \neq v_y = 0 \qquad v_x = 0 \neq -u_y = -2y.$$

Thus, the Cauchy-Riemann equations only hold at $(x,y) = (0,0)$ and as we saw previously that this function is only differentiable at $z = 0$ and nowhere else.

We can generalize the Cauchy-Riemann equations further to second order. Suppose we have $f = u + iv$. Then

$$u_{xx} = \frac{\partial}{\partial x} u_x = \frac{\partial}{\partial x} v_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y} v = \frac{\partial}{\partial y}\frac{\partial}{\partial x} v = \frac{\partial}{\partial y} v_x = \frac{\partial}{\partial y}(-u_y) = -u_{yy}.$$

Thus, we have

$$u_{xx} + u_{yy} = 0, \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, we also have

$$v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = -(u_x)_y = -(v_y)_y = -v_{yy},$$

which gives

$$v_{xx} + v_{yy} = 0, \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

These are the *Laplace's equations* in 2D we saw earlier.

For $f : \mathbb{R} \to \mathbb{R}$, we know that $f'(x) = 0$ implies that f is constant. But for $f : \mathbb{C} \to \mathbb{C}$, we can use the Cauchy-Riemann equations. Since $f'(z) = \frac{\partial f}{\partial x}$,

$$f'(z) = 0 \implies u_x + iv_x = 0 \implies u_x = 0, v_x = 0.$$

By Cauchy-Riemann, we also have $u_y = v_y = 0$. Since $u_x = 0$, we know that for fixed y, $u(x,y)$ is some constant that could depend on y. Thus, we have

$$u(x,y) = g(y).$$

But $u_y = 0$, so $g'(y) = 0$, which means g is actually a constant independent of y. Thus, u is globally constant. Similar argument applies to v as well.

Chapter 4

Möbius Transformation

Definition 4.0.1 (Möbius transformation)**.** A *Möbius transformation* is a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$.

Remark. If $ad = bc$, then $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$, so rows are linearly dependent: $\lambda(a,b) + \mu(c,d) = (0,0)$, which implies that

$$a = \frac{-\mu}{\lambda} c \qquad b = \frac{-\mu}{\lambda} d.$$

Then

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{-\frac{\mu}{\lambda}(cz+d)}{cz+d} \\ &= -\frac{\mu}{\lambda}, \end{aligned}$$

which is a constant independent of z.

Proposition 4.0.2 (Composite Möbius transforms is Möbius)**.** *If $f_1(z), f_2(z)$ are Möbius transforms, then then $f_1(f_2(z))$ is also a Möbius transform.*

Proof. Suppose

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Then

$$\begin{aligned} f_1(f_2(z)) &= \frac{a_1 \frac{a_2 z+b_2}{c_2 z+d_2} + b_1}{c_1 \frac{a_2 z+b_2}{c_2 z+d_2} + d_1} \\ &= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)}, \end{aligned}$$

which is another Möbius transform. $\square$

Remark. Note that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

and the entries coincide with the composite Möbius transform. If we denote $f_M(z)$ to be a transform associated with a 2×2 matrix M, then we have just shown that

$$f_M(f_N(z)) = f_{MN}(z).$$

Remark. Since $f_I(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z$, the inverse of f_M is $f_{M^{-1}}$.

Remark. Note that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies f_M = \frac{az+b}{cz+d}.$$

Meanwhile,

$$\lambda M = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \implies f_{\lambda M} = \frac{\lambda a z + \lambda b}{\lambda c z + \lambda d} = \frac{az+b}{cz+d} = f_M.$$

Thus, scaling the matrices doesn't affect the resulting Möbius tranformation.

4.1 Inverse of Möbius transformation

Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since the scaling part is redundant, we simply ignore it and obtain the inverse Möbius transform as follows:

$$f(z) = \frac{az+b}{cz+d} \implies f^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Remark. Since Möbius transforms have inverses, they should be bijections. However, some details should be noted. If $c \neq 0$, then $\frac{az+b}{cz+d}$ is undefined at $z = -\frac{d}{c}$.

Let's consider the value at $z = -\frac{d}{c}$ to be infinity. It turns out that we can evaluate $\frac{az+b}{cz+d}$ at ∞:

$$\begin{aligned}\lim_{z\to\infty} \frac{az+b}{cz+d} &= \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} \\ &= \frac{a}{c}.\end{aligned}$$

When $c = 0$, we view $\frac{a}{c}$ as ∞. So now we view Möbius transformations as functions from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$. This makes all Möbius transformations into bijections. Here, we call $\mathbb{C} \cup \{\infty\}$ the *extended complex plane* (also called *Riemann sphere*).

Remark. For real functions, there are multiple notions of going to infinity: $x \to +\infty$ and $x \to -\infty$. But for complex functions, we work with only one infinite point.

Fact. If we apply a Möbius transformation to a line or a circle in the complex plane, we would get a line or a circle again (circles can turn into lines and vice versa).

Example 4.1.1. Consider $f(z) = \frac{z-1}{iz+i}$, let's apply this to the unit circle, i.e. take $z = e^{i\theta}$. Then

$$\begin{aligned}f(e^{i\theta}) = \frac{e^{i\theta}-1}{i(e^{i\theta}+1)} &= \frac{\cos\theta - 1 + i\sin\theta}{i(\cos\theta + 1 + i\sin\theta)} \\ &= \frac{-2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{i\left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)} \\ &= \frac{2i(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})\sin\frac{\theta}{2}}{2i\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\cos\frac{\theta}{2}} \\ &= \tan\frac{\theta}{2}.\end{aligned}$$

Note that $\theta \in (-\pi, \pi)$ and we have $\tan -\frac{\pi}{2} = -\infty$ and $\tan\frac{\pi}{2} = -+\infty$. We have mapped a unit circle to a line (real line).

Fact. f sends the interior of the unit disk to the interior of the upper half-plane. If $g(z)$ is holomorphic on the upper half-plane, then $g(f(z))$ is a holomorphic function on the unit disk. Taking real and imaginary parts gives a solution to the Laplace equation.

Remark. Stereographic projection φ is a bijection that maps a sphere to the extended complex plane. It doesn't preserve distance, but it preserves functions being holomorphic.

Proposition 4.1.2. *Suppose $f(z) = \frac{az+b}{cz+d}$ is a Möbius transformation. If $c = 0$ then*

$$f(z) = \frac{a}{d}z + \frac{b}{d},$$

and if $c \neq 0$, then

$$f(z) = \frac{bc - ad}{c^2} \frac{1}{z + \frac{d}{c}} + \frac{a}{c}.$$

In particular, every Möbius transformation is a composition of translations, dilations, and inversions.

Proof. Simplify. $\square$

Theorem 4.1.3. *Möbius transformations map circles and lines into circles and lines.*

Proof. Translations and dilations certainly map circles and lines into circles and lines, so by the previous proposition, we only have to prove the statement of the theorem for the inversion $f(z) = \frac{1}{z}$.

The equation for a circle centered at $x_0 + iy_0$ with radius r is $(x - x_0)^2 + (y - y_0)^2 = r^2$, which we can transform to

$$\alpha \left(x^2 + y^2\right) + \beta x + \gamma y + \delta = 0$$

for some real numbers α, β, γ, and δ that satisfy $\beta^2 + \gamma^2 > 4\alpha\delta$. The above expression is more convenient for us, because it includes the possibility that the equation describes a line (precisely when $\alpha = 0$).

Suppose $z = x + iy$ satisfies the above expression; we need to prove that $u + iv := \frac{1}{z}$ satisfies a similar equation. Since

$$u + iv = \frac{x - iy}{x^2 + y^2},$$

we can rewrite the transformed equation as

$$\begin{aligned} 0 &= \alpha + \beta \frac{x}{x^2 + y^2} + \gamma \frac{y}{x^2 + y^2} + \frac{\delta}{x^2 + y^2} \\ &= \alpha + \beta u - \gamma v + \delta \left(u^2 + v^2\right). \end{aligned}$$

But this equation says that $u + iv$ lies on a circle or line. $\square$

Fact. The stereographic projection of a circle on the sphere (intersection of a plane and a sphere) is a circle in the plane. Möbius transformations take circles on the sphere to other circles of the sphere (some of these stereographically project to lines in the plane).

Chapter 5

Exponential,Trigonometric, and Logarithmic Functions

5.1 Exponential Functions

$$e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y \implies e^z = u(x,y) + iv(x,y)$$

where

$$u(x,y) = e^x \cos y$$
$$v(x,y) = e^x \sin y$$

5.2 Trigonometric Functions

For $z \in \mathbb{C}$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Remark. $\sin z, \cos z$ are holomorphic since e^z is holomorphic and so is e^{iz} and e^{-iz}.

Trigonometric identities hold for complex numbers.

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(2z) = 2 \sin z \cos z$$

5.3 Logarithmic Functions

We want $\log z$ to be the unique inverse to the exponential function, i.e. we want $e^{\log z} = z$, but then we would also have

$$e^{\log z + 2\pi i k} = z.$$

Definition 5.3.1 (Principal logarithm)**.** The *principal logarithm* is the function defined by

$$\log\left(re^{i\theta}\right) = \log r + i\theta.$$

where $-\pi < \theta \leq \pi$.

Let's check if $\log z$ is differentiable. If

$$z = x + iy = re^{i\theta},$$

then $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ when $x \neq 0$.

$$\begin{aligned} \log x + iy &= \frac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1}\left(\frac{y}{x}\right) \\ &= u(x, y) + iv(x, y). \end{aligned}$$

Then

$$\begin{aligned} u_x &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2} \\ u_y &= \frac{1}{2} \cdot \frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2} \\ v_x &= \frac{-y}{x^2} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} = -\frac{y}{x^2 + y^2} \\ v_y &= \frac{1}{x} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}. \end{aligned}$$

Thus, we see that the Cauchy-Riemann equations hold for logarithms.

Chapter 6

Complex Integration

6.1 Definition and Basic Properties

If $f : \mathbb{R} \to \mathbb{C}$, define

$$\int_a^b f(x)dx = \int_a^b \Re f(x)dx + i \int_a^b \Im f(x)dx.$$

Question. But how to integrate a function $f : \mathbb{C} \to \mathbb{C}$?

For real functions, going from a point $\gamma(a)$ to $\gamma(b)$ can only happen one way (follow the real axis) but in $\mathbb{C}$, we will have to specify the path from $\gamma(a)$ to $\gamma(b)$.

Definition 6.1.1 (Path/curve)**.** A *path/curve* is the image of a function $\gamma : [a, b] \to \mathbb{C}$.

Definition 6.1.2 (Integral)**.** The *integral* of the function $f : \mathbb{C} \to \mathbb{C}$ along the path parametrized by $\gamma : [a, b] \to \mathbb{C}$ is

$$\int_a^b f(\gamma(t))\gamma'(t)dt.$$

This is the integral of a function $\mathbb{R} \to \mathbb{C}$, so we already have a definition for it.

Aside,

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx.$$

Suppose we have a different parametrization of the image of $\gamma(t)$. Write this parametrization as $\gamma(\theta(t))$ where $\theta : [a, b] \to [a, b]$ is a continuous reparametrization of the interval $[a, b]$ satisfying $\theta(a) = a$, $\theta(b) = b$ and θ is increasing. Then

$$\int_a^b f(\gamma(\theta(t)))\gamma'(\theta(t))\theta'(t)dt = \int_{\theta(a)}^{\theta(b)} f(\gamma(u))\gamma'(u)du$$

where $u = \theta(t)$ and $du = \theta'(t)dt$. So the integrals for $\gamma(\theta(t))$ and $\gamma(t)$ are the same, thus the integral depends on the curve in $\mathbb{C}$, not how we parametrize it.

We will use

$$\oint_\gamma f(z)dz$$

to denote the integral.

Example 6.1.3. If $\gamma(t) = t$, then $\gamma'(t) = 1$ and

$$\oint_\gamma f(z)dz = \int_a^b f(t)dt.$$

Example 6.1.4. If $\gamma(t) = t + it^2$ and $f(z) = 1$, then $\gamma'(t) = 1 + 2it$ and

$$\begin{aligned}
\int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b (1+2it)dt \\
&= \int_a^b 1dt + i\int_a^b 2tdt \\
&= b - a + i(b^2 - a^2) \\
&= (b + ib^2) - (a + ia^2) \\
&= \gamma(b) - \gamma(a).
\end{aligned}$$

Example 6.1.5 (Very important example)**.** Consider $\gamma(t) = e^{it}$ where $0 \le t \le 2\pi$. So $\gamma(t)$ is the counterclockwise unit circular path. If $f(z) = z^n$ for some $n \in \mathbb{Z}$. Then

$$\begin{aligned}
\oint_\gamma f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \\
&= \int_0^{2\pi} e^{int} \cdot ie^{it}dt \\
&= i\int_0^{2\pi} e^{it(n+1)}dt.
\end{aligned}$$

If $n \neq -1$, then $n + 1 \neq 0$, the integral evaluates to

$$\begin{aligned}
i\left.\frac{e^{it(n+1)}}{i(n+1)}\right|_{t=0}^{2\pi} &= i\left(\frac{1}{i(n+1)} - \frac{1}{i(n+1)}\right) \\
&= 0.
\end{aligned}$$

If $n+1=0$, then $n=-1$ and so

$$\begin{aligned} i\int_0^{2\pi} e^{it(n+1)}dt &= i\int_0^{2\pi} 1dt \\ &= 2\pi i, \end{aligned}$$

which is not zero.

Chapter 7

Complex Integration (Cont'd)

7.1 Basic Properties

(i) If $\mu, \lambda \in \mathbb{C}$, then

$$\begin{aligned}\oint_\gamma \lambda f(z) + \mu g(z) dz &= \int_a^b (\lambda f(\gamma(t) + \mu g(\gamma(t))\gamma'(t) dt \\ &= \lambda \int_a^b f(\gamma(t)\gamma'(t) dt + \mu \int_a^b g(\gamma(t))\gamma'(t) dt \\ &= \lambda \oint_\gamma f(z) dz + \mu \oint_\gamma g(z) dz.\end{aligned}$$

(ii)

$$\int_{\gamma_1\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

(iii)

$$\int_{-\gamma} f = -\oint$$

(iv)

$$\left|\oint_\gamma f\right| \leq \max{}_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma),$$

where

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

View $|\gamma'(t)|$ as the speed a particle is travelling at and $\gamma(t)$ as the position of that particle at time t. Then integrating it gives the total distance.

(v) **(Triangle Inequality)**

$$\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b \left|f(\gamma(t))\gamma'(t)\right| dt$$

(vi) **(ML-Lemma)**

$$\begin{aligned}\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| &\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|dt \\ &= ML\end{aligned}$$

where $M = \max_{a\leq t\leq b}|f(\gamma(t))|$ and $L = \int_a^b |\gamma'(t)|dt$.

7.1.1 Antiderivatives

Theorem 7.1.1 (Fundamental Theorem of Calculus)**.** *If F is holomorphic on some subset $G \subseteq \mathbb{C}$ and $\frac{d}{dz}F(z) = f(z)$. Then*

$$\oint_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Let $F(x+iy) = u(x,y) + iv(x,y)$ and $\gamma(t) = \alpha(t) + i\beta(t)$. Then

$$F(\gamma(t)) = u(\alpha(t), \beta(t)) + iv(\alpha(t), \beta(t)).$$

By chain rule,

$$\begin{aligned}\frac{d}{dt}F(\gamma(t)) &= u_x\alpha'(t) + u_y\beta'(t) + iv_x\alpha'(t) + iv_y\beta'(t) \\ &= u_x(\alpha'(t) + i\beta'(t)) + iv_x(\alpha'(t) + i\beta'(t)) \qquad (u_x = v_y \text{ by CR}) \\ &= F(\gamma(t))\gamma'(t).\end{aligned}$$

□

Definition 7.1.2 (Closed curve)**.** A *closed curve* is a curve where the start and end points are the same, i.e. $\gamma(a) = \gamma(b)$.

So if $f(z) = \frac{d}{dz}F(z)$, the integral of $f(z)$ around a closed curve is zero:

$$\oint_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

Example 7.1.3. Let γ be the path of unit circle counterclockwise. Then

$$\oint_\gamma \frac{1}{z} dz = 2\pi i$$

which is not zero, implying that there is no holomorphic function $F(z)$ defined on the whole unit circle, having derivative $\frac{1}{z}$.

However, consider the principal logarithm $\log(re^{i\theta}) = \log(r) + i\theta$, we have

$$\frac{d}{dz} \log(x+iy) = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}.$$

So $\frac{1}{z}$ does have an antiderivative on $\mathbb{C} \backslash \mathbb{R}_{\leq 0}$.

It turns out that if $f(z)$ is continuous and $\oint_\gamma f(z)dz = 0$ for any closed curve, then $f(z)$ has an antiderivative, i.e. there's $F(z)$ such that $F'(z) = f(z)$.

We know that by fundamental theorem of calculus

$$\frac{d}{dx} \int_a^x f(t)dt = f(x).$$

By analogy, we want

$$F(w) = \oint_{(} z)dz$$

where γ is a curve from a fixed basepoint q to w.

First let's check that this doesn't depend on the choice of path from q to w. Suppose δ_1, δ_2 are two paths from q to w. Observe that the reverse of δ_2 is a curve from w to q, and the path obtained by following δ_1, then the reverse of δ_2 goes from q to w to q, so it is a closed curve.

Write $\delta_1 - \delta_2$ for the closed curve above. Then by assumption, we have

$$\int_{\delta_1 - \delta_2} f(z)dz = 0.$$

This implies that

$$\int_{\delta_1} f(z)dz + \int_{-\delta_2} f(z)dz = \int_{\delta_1} f(z)dz - \int_{\delta_2} f(z)dz = 0.$$

Hence,

$$\int_{\delta_1} f(z)dz = \int_{\delta_2} f(z)dz.$$

Thus, the choice of path doesn't matter and so the formula $F(w) = \oint_\gamma f(z)dz$ where γ is any path from q to w makes sense. Let's now check $\frac{d}{dw}F(w) = f(w)$.

$$\frac{d}{dw}F(w) = \lim_{h\to 0}\frac{F(w+h)-F(w)}{h}.$$

To evaluate $F(w+h)$, we can choose the path of integration from q to $w+h$ arbitrarily. Let's choose one that goes from q to w then to $w+h$ along a line segment (only from w to $w+h$). This line segment has length $|h|$.

If our function is holomorphic at a point w, it is differentiable on a disk $D(w,\epsilon)$ for some $\epsilon > 0$. So if $|h| < \epsilon$, then the line segment ℓ from w to $w+h$ is contained in $D(w,\epsilon)$ and hence in a region where the function is differentiable.

Now $F(w+h)-F(w)$ is simply the integral of $f(z)$ from w to $w+h$. We want

$$\lim_{h\to\infty}\frac{1}{h}\int_\ell f(z)dz - f(w) = 0.$$

Note that $f(w)$ is a constant independent of z. Thus,

$$\int_\ell f(w)dz = f(w)\int_\ell 1dz = f(w)h.$$

$$\lim_{h\to\infty}\frac{\int_\ell f(z)dz - \int_\ell f(w)dz}{h} = \lim_{h\to\infty}\frac{\int_\ell (f(z)-f(w))dz}{h}.$$

By ML-lemma,

$$\begin{aligned}\left|\frac{\int_\ell (f(z)-f(w))dz}{h}\right| &= \frac{\left|\int_\ell (f(z)-f(w))dz\right|}{|h|}\\ &\le \max{}_{z\in\ell}|f(z)-f(w)|\cdot\frac{\text{length}(\ell)}{|h|}\\ &= \max{}_{z\in\ell}|f(z)-f(w)|.\end{aligned}$$

So it suffices to show that

$$\lim_{h\to 0}\max{}_{z\in\ell}|f(z)-f(w)| = 0.$$

Since $f(z)$ is continuous at w, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|z-w| < \delta$ implies $|f(z)-f(w)| < \epsilon$. So when $|h| < \delta$, any $z \in \ell$ obeys $|z-w| < \delta$. Then also $|f(z)-f(w)| < \epsilon$. So for $|h| < \delta$,

$$\left|\frac{F(w+h)-F(w)}{h} - f(w)\right| < \epsilon.$$

Hence,

$$\lim_{h\to 0} \frac{F(w+h)-F(w)}{h} = f(w).$$

as needed.

7.2 Cauchy's Theorem

Question. How do we check that $\oint_\gamma f(z)dz = 0$ for any closed curve γ?

Theorem 7.2.1 (Cauchy's Theorem)**.** *Suppose $\gamma : [a,b] \to \mathbb{C}$ is a closed curve and $f(z)$ is holomorphic on γ and in the region enclosed by the curve γ. Then*

$$\oint_\gamma f(z)dz = 0.$$

Proof. Recall that a vector field is a function $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\vec{F}(x,y) = (F_1(x,y), F_2(x,y)).$$

The line integral of $\vec{F}$ along a curve γ is

$$\oint_\gamma \vec{F} \cdot d\vec{\ell} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)dt$$

where $\gamma(t) = (\alpha(t), \beta(t))$ and $\gamma'(t) = (\alpha'(t), \beta'(t))$ and so

$$\oint_\gamma \vec{F} \cdot \vec{\ell} = \int_a^b F_1(\alpha(t), \beta(t))\alpha'(t) + F_2(\alpha(t), \beta(t))\beta'(t)dt.$$

Now recall the Stokes' theorem

$$\oint_\gamma \vec{F} \cdot d\vec{\ell} = \int_{\text{region enclosed by } \gamma} \vec{\nabla} \times \vec{F} dA$$

where

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_2}{\partial dx} - \frac{\partial F_1}{\partial y}.$$

Then

$$\oint_\gamma (\vec{F} \cdot \hat{n})d\ell = \int_a^b F_1(\alpha(t), \beta(t))\beta'(t) + F_2(\alpha(t), \beta(t))(-\alpha'(t))dt.$$

The divergence theorem says that

$$\oint_\gamma (\vec{F} \cdot \hat{n})d\ell = \int_{\text{area enclosed by } \gamma} \vec{\nabla} \cdot \vec{F} dA$$

where

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

Let $f(x+iy) = u(x,y) + iv(x,y)$ and $\gamma(t) = \alpha(t) + i\beta(t)$.

$$\begin{aligned}\oint_\gamma f(z)dz &= \int_a^b (u+iv)(\alpha'(t) + i\beta'(t))dt \\ &= \int_a^b u\alpha'(t) - v\beta'(t)dt + i\int_a^b v\alpha'(t) + u\beta'(t)dt.\end{aligned}$$

Note that $u' - v\beta' = (u, -v)\cdot(\alpha', \beta')$ and $v\alpha' + u\beta' = (u, -v)\cdot(\beta', \alpha')$. Now let $\vec{F}(x,y) = (u(x,y), -v(x,y))$. Then

$$\oint_\gamma f(z)dz = \oint_\gamma \vec{F}\cdot d\vec{\ell} + i\oint_\gamma (\vec{F}\cdot\hat{n})d\ell.$$

$$\begin{aligned}\vec{\nabla}\times\vec{F} &= \frac{\partial - v}{\partial x} - \frac{\partial u}{\partial y} = -v_x - u_y = 0 \qquad (u_y = -v_x \text{ by CR}), \\ \vec{\nabla}\cdot\vec{F} &= \frac{\partial u}{\partial x} + \frac{\partial - v}{\partial y} = u_x - v_y = 0 \qquad (u_x = v_y \text{ by CR}).\end{aligned}$$

So by Stokes' theorem and the divergence theorem

$$\oint_\gamma f(z)dz = \int_{\text{area enclosed by }\gamma} 0dA + i\int_{\text{area enclosed by }\gamma} 0dA = 0 + i0 = 0.$$

$\square$

Remark. The example of $\oint_\gamma \frac{1}{z}dz = 2\pi i$ does not contradict the Cauchy's theorem because $\frac{1}{z}$ is not holomorphic at $z = 0$.

Remark. Checking that closed curves have well-defined interior regions is not a triviality: it is the content of the Jordan Curve Theorem (out of scope).

Remark. There are several formulations of Cauchy's Theorem. We will assume that $f'(z)$ is continuous. Some formulations remove the concept of interior region and instead use the notion of a homotopy.

7.2.1 Cauchy Integral

Theorem 7.2.2 (Cauchy integral formula (1st version)). *Suppose $f(z)$ is holomorphic on the closed disk of radius R centered at $a \in \mathbb{C}$. Then*

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

where γ is the anticlockwise circle of radius R centered at a.

Proof. $\frac{f(z)}{z-a}$ may not be holomorphic at $z = a$, so to apply Cauchy's theorem we need a curve that doesn't enclose a. We can create such curve by traversing a donut-like path obtained by traversing a clockwise small circle with radius r centered at a after we reached the endpoint of the original curve and then traverse back to the endpoint. Then the enclosed area will not include a.

This curve of integration has 4 parts:

1. γ_1 : big circle of radius R, anticlockwise,
2. γ_2 : line segment connecting from the big circle to the small circle,
3. γ_3 : small circle of radius r, clockwise,
4. γ_4 : line segment connecting from the small circle to the big circle.

Note that the integrations of the two line segments cancel out each other. Then Cauchy's theorem tells us that

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} = 0 \implies \int_{\gamma_1} + \int_{\gamma_3} = 0.$$

Thus,

$$\int_{\gamma_1} = -\int_{\gamma_3} = \int_{-\gamma_3},$$

which implies that the integral of the big circle anticlockwise is equal to the integral of the small circle anticlockwise. Hence,

$$\int_{\gamma_1} \frac{f(z)}{z-a} dz = \int_{-\gamma_3} \frac{f(z)}{z-a} dz.$$

for any $r < R$, and so we can take $r \to 0$. Now we want to show that

$$\int_{\gamma_3} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Let $\gamma(t) = a + re^{it}$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\oint_\gamma \frac{f(a)}{z-a}dz &= f(a)\oint_\gamma \frac{1}{z-a}dz \\ &= f(a)\int_0^{2\pi} \frac{1}{\gamma(t)-a}\gamma'(t)dt \\ &= f(a)\int_0^{2\pi} \frac{1}{re^{it}}ire^{it}dt \\ &= 2\pi i f(a).\end{aligned}$$

So we want

$$\oint_\gamma \frac{f(z)}{z-a}dz = \oint_\gamma \frac{f(a)}{z-a}dz,$$

i.e.,

$$\oint_\gamma \frac{f(z)-f(a)}{z-a}dz = 0.$$

To show this, we apply the ML-lemma:

$$\left|\oint_\gamma \frac{f(z)-f(a)}{z-a}dz\right| \le \max{}_{z\in\gamma}\left|\frac{f(z)-f(a)}{z-a}\right| \cdot \mathrm{length}(\gamma)$$

where γ is all points at distance r from a. Then $z \in \gamma \implies |z-a| = r$ and $\mathrm{length}(\gamma) = 2\pi r$. Thus, we get

$$\begin{aligned}\left|\oint_\gamma \frac{f(z)-f(a)}{z-a}dz\right| &\le \max{}_{z\in\gamma}\frac{|f(z)-f(a)|}{r} \cdot 2\pi r \\ &= 2\pi \cdot \max{}_{z\in\gamma}|f(z)-f(a)|.\end{aligned}$$

Since $f(z)$ is differentiable, it is continuous. So fro any $\epsilon > 0$. there is a $\delta > 0$ such that $|z-a| = r < \delta \implies |f(z)-f(a)| < \epsilon$. So by taking $r < \delta$, we get

$$\left|\oint_\gamma \frac{f(z)-f(a)}{z-a}dz\right| < 2\pi\epsilon$$

for any $\epsilon > 0$, which implies that the absolute value must be zero. Hence,

$$\oint_\gamma \frac{f(z)-f(a)}{z-a}dz = 0,$$

and this implies that

$$\oint_\gamma \frac{f(z)}{z-a}dz = \oint_\gamma \frac{f(a)}{z-a}dz = 2\pi i f(a).$$

$\square$

Theorem 7.2.3 (Cauchy integral formula (2nd version)). *Let γ be a closed curve that encloses $a \in \mathbb{C}$ exactly once anticlockwise. Suppose $f(z)$ is holomorphic inside γ. Then*

$$\oint_\gamma \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Proof. Similar proof to previous one. $\square$

Example 7.2.4. Suppose $\omega \geq 0$ is a real number. Then

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2+1} dx = \pi e^{-\omega}.$$

Consider

$$f(z) = \frac{e^{i\omega z}}{z^2+1} = \frac{\frac{e^{i\omega z}}{z+i}}{z-i} = \frac{g(z)}{z-i}$$

where $g(z) = \frac{e^{i\omega z}}{z+i}$. Consider the integral of $f(z)$ over the semi-circle curve anticlockwise with radius R consisting of a line segment ℓ. On the line segment, let $\delta : [-R, R] \to \mathbb{C}$ be defined as $\delta(t) = t$. Then

$$\begin{aligned}
\int_\delta \frac{g(z)}{z-i} dz &= \int_{-R}^{R} \frac{g(\delta(t))}{\delta(t)-i} \delta'(t) dt \\
&= \int_{-R}^{R} \frac{g(t)}{t-i} dt \\
&= \int_{-R}^{R} \frac{e^{i\omega x}}{x^2+1} dx \\
&= \int_{-R}^{R} \frac{\cos(\omega x)}{x^2+1} dx + i \int_{-R}^{R} \frac{\sin(\omega x)}{x^2+1} dx.
\end{aligned}$$

Our goal is to compute the real part of the above expression. So what we want is

$$\lim_{R\to\infty} \int_\ell = \lim_{R\to\infty} \oint_\gamma - \int_{\text{arc}}.$$

By Cauchy integral formula, we have

$$\begin{aligned}
\lim_{R\to\infty} \oint_\gamma \frac{g(z)}{z-i} dz &= 2\pi i g(i) \\
&= 2\pi i \frac{e^{i\omega i}}{i+i} \\
&= \pi e^{-\omega}.
\end{aligned}$$

Now we compute

$$\lim_{R\to\infty}\int_{\text{arc}}\frac{e^{i\omega z}}{z^2+1}dz.$$

By ML-inequality, we have

$$\left|\int_{\text{arc}}\right| \leq \max_{z\in\text{arc}}\left|\frac{e^{i\omega z}}{z^2+1}\right|\cdot\text{length(arc)}.$$

We needed to work with the upper half plane so that for $z \in \text{arc}$, $\Im(z) \geq 0$, so then because $\omega \geq 0$, $\Im(\omega z) \geq 0$. If $\omega z = a + ib$, then $b \geq 0$ and $i\omega z = -b + ia$ has non-positive real part. So

$$|e^{i\omega z}| = e^{\Re(i\omega z)} \leq e^0 = 1.$$

Also $z \in \text{arc} \implies |z| = R$ and so $|z^2| = R^2$. Thus we have $|z^2+1| \geq R^2 - 1$. Hence,

$$\frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}.$$

So ML-inequality becomes

$$\left|\int_{\text{arc}}\right| = \frac{1}{R^2-1}\pi R \to 0 \qquad \text{as R} \to \infty.$$

So as $R \to \infty$,

$$\int_{\text{arc}} \to 0.$$

Hence, we have

$$\begin{aligned}\oint_\gamma - \int_{\text{arc}} &= \int_\ell \\ \pi e^{-\omega} - 0 &= \int_{-\infty}^{\infty}\frac{\cos(\omega x)}{x^2+1}dx + i\int_{-\infty}^{\infty}\frac{\sin(\omega x)}{x^2+1}dx.\end{aligned}$$

Note that

$$\oint_\gamma \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i f(a) & \text{if } a \text{ is in the region enclosed by } \gamma \\ 0 & \text{else.}\end{cases}$$

So if $\gamma : [a', b'] \to \mathbb{C}$ is the curve,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}dz \\ &= \frac{1}{2\pi i}\int_{a'}^{b'}\frac{f(\gamma(t)}{\gamma(t)-a}\gamma'(t)dt.\end{aligned}$$

So for example, if $f(z)$ is zero on the unit circle: $f(e^{i\theta}) = 0$, then for $|a| < 1$,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - a} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} 0 d\theta \\ &= 0. \end{aligned}$$

So if $f(z)$ is zero on the circle, it is also zero inside. For comparison,

$$f(x + iy) = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

is real differentiable and is zero on the unit circle, but not insides, so it's not holomorphic.

Claim.

$$f'(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - a)^2} dz.$$

Proof.

$$\begin{aligned} 2\pi i f'(a) &= \lim_{h \to 0} 2\pi i \left(\frac{f(a + h) - f(a)}{h} \right) \\ &= \lim_{h \to 0} \frac{\oint_\gamma \frac{f(z)}{z - (a+h)} dz - \oint_\gamma \frac{f(z)}{z - a} dz}{h}. \end{aligned}$$

We assumed that a is inside γ, but we should also check $a + h$. Let $\gamma : [a', b'] \to \mathbb{C}$ be the curve. Consider $|\gamma(t) - a|$ (distance from $\gamma(t)$ to a).

The domain of $|\gamma(t) - a|$ is $[a', b']$, a compact set (closed and bounded by Heine-Borel). Continuous image of a compact set is also compact, so in particular closed and so the complement is open. Note that $|\gamma(t) - a| \neq 0$ (otherwise $\gamma(t) = a$, which means that a would be on the curve γ, which we don't allow). Then 0 is in the complement of the image of $|\gamma(t) - a|$. But since the set is open, it must also contain a neighborhood of 0. We can assume it is of the form $(-\epsilon, \epsilon)$.

Conclusion: not only does $\gamma(t)$ avoid a, it never comes within ϵ of it: $|\gamma(t) - a| \geq \epsilon$ (complement contains $[0, \epsilon)$). If $|h| < \epsilon$, then also $a + h$ is inside $\gamma(t)$.

So for h small enough, we can write

$$
\begin{aligned}
2\pi i\frac{f(a+h)-f(a)}{h} &= \frac{\oint_\gamma \frac{f(z)}{z-(a+h)} - \frac{f(z)}{z-a}dz}{h} \\
&= \frac{1}{h}\oint_\gamma \frac{(z-a)f(z)-(z-a-h)f(z)}{(z-a)(z-a-h)}dz \\
&= \frac{1}{h}\oint_\gamma \frac{hf(z)}{(z-a)(z-a-h)}dz \\
&= \oint_\gamma \frac{f(z)}{(z-a)(z-a-h)}dz.
\end{aligned}
$$

We want

$$\lim_{h\to 0}\oint_\gamma \frac{f(z)}{(z-a)(z-a-h)}dz = \oint_\gamma \frac{f(z)}{(z-a)^2}dz.$$

or equivalently,

$$\lim_{h\to 0}\oint_\gamma \left(\frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2}\right)dz = 0.$$

$$
\begin{aligned}
\lim_{h\to 0}\oint_\gamma \left(\frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2}\right)dz &= \lim_{h\to 0}\oint_\gamma \frac{f(z)(z-a)-f(z)(z-a-h)}{(z-a)^2(z-a-h)}dz \\
&= \lim_{h\to 0} h\oint_\gamma \frac{f(z)}{(z-a)^2(z-a-h)}dz.
\end{aligned}
$$

The ML-inequality says

$$\left|\oint_\gamma\right| \le \max{}_{z\in\gamma}\frac{|f(z)|}{|z-a|^2|z-a-h|}\cdot \text{length}(\gamma).$$

Notice that $|z-a|\ge\epsilon$ so for $|h|\le\frac{\epsilon}{2}$.

$$
\begin{aligned}
|z-a-h| &\ge |z-a|-|h| \\
&\ge \epsilon - \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2}.
\end{aligned}
$$

Hence,

$$\left|\oint_\gamma\right| \le \max{}_{z\in\gamma}\frac{|f(z)|}{\epsilon^2\cdot\frac{\epsilon}{2}}\cdot \text{length}(\gamma).$$

The bound is independent of h. $\square$

7.3 Liouville's Theorem

Theorem 7.3.1 (Liouville's Theorem)**.** *Suppose $f(z)$ is holomorphic on all of $\mathbb{C}$, and $f(z)$ is bounded, i.e., $|f(z)| \leq M$ for some fixed M. Then $f(z)$ is constant.*

Remark. Note that $f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1} = 1 - \frac{2}{x^2+y^2+1}$ is real differentiable and bounded but it's not constant. So the theorem implies it's not holomorphic.

Proof. Let's compute $f'(a)$ for some $a \in \mathbb{C}$. Let γ be the circle of radius R centered at a. Then

$$f'(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^2} dz.$$

ML-lemma says

$$\begin{aligned}
|f'(a)| &\leq \frac{1}{2\pi} \max{}_{z\in\gamma} \frac{|f(z)|}{|z-a|^2} \cdot \text{length}(\gamma) \\
&= \frac{1}{2\pi} \max{}_{z\in\gamma} \frac{|f(z)|}{R^2} \cdot 2\pi R \\
&\leq \frac{1}{2\pi} \frac{M}{R^2} 2\pi R \\
&= \frac{M}{R}.
\end{aligned}$$

Since M doesn't depend on R,

$$|f'(a)| \leq \frac{M}{R} \qquad \text{for any } R > 0.$$

As $R \to \infty$, this gets arbitrarily small. So $|f'(a)| = 0$ and hence $f'(a) = 0$. So it must be true that $f(a)$ is constant. ☐

Claim. Suppose $f(z)$ is holomorphic inside γ and a is insides γ. Then $f''(a)$ eixsts and

$$f''(a) = \frac{1}{\pi i} \oint_\gamma \frac{f(z)}{(z-a)^3} dz.$$

Proof. For w inside γ, let

$$f'(w) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-w)^2} dz.$$

Then

$$\begin{aligned}2\pi i \cdot \frac{f'(a+h)-f'(a)}{h} &= \frac{1}{h}\left(\oint_\gamma \frac{f(z)}{(z-a-h)^2}dz - \oint_\gamma \frac{f(z)}{(z-a)^2}dz\right)\\ &= \frac{1}{h}\oint_(z)\cdot \frac{(z-a)^2-(z-a-h)^2}{(z-a)^2(z-a-h)^2}dz\\ &= \frac{1}{h}\oint_\gamma \frac{h(2z-2a-h)}{(z-a)^2(z-a-h)^2}\cdot f(z)dz.\end{aligned}$$

Then to show that

$$\lim_{h\to 0}\oint_\gamma \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2}\cdot f(z)dz = 2\oint_\gamma \frac{f(z)}{(z-a)^3}dz,$$

we show

$$\lim_{h\to 0}\oint_\gamma \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2}\cdot f(z)dz - \oint_\gamma \frac{2f(z)}{(z-a)^3}dz = 0$$

The LHS becomes

$$\begin{aligned}&\lim_{h\to 0}\oint_\gamma \frac{(z-a)(2(z-a)-h)-2(z-a-h)^2}{(z-a)^3(z-a-h)^2}\cdot f(z)dz\\ &= \lim_{h\to 0}\oint_\gamma \frac{2(z-a)^2-h(z-a)-2((z-a)^2-2h(z-a)+h^2)}{(z-a)^3(z-a-h)^2}\cdot f(z)dz\\ &= \lim_{h\to 0}\oint_\gamma \frac{3(z-a)h-2h^2}{(z-a)^3(z-a-h)^2}\cdot f(z)dz\\ &= \lim_{h\to 0} 3h\oint_\gamma \frac{(z-a)f(z)}{(z-a)^3(z-a-h)^2}dz - 2h^2\oint_\gamma \frac{f(z)}{(z-a)^3(z-a-h)^2}dz.\end{aligned}$$

For the limit as $h\to\infty$ to exist and be zero, it's enough that the integral remain bounded.

$$\begin{aligned}\left|\oint_\gamma \frac{(z-a)}{(z-a)^3(z-a-h)^2}f(z)dz\right| &\le \max{}_{z\in\gamma}\left|\frac{f(z)}{(z-a)^2(z-a-h)^2}\right|\text{length}(\gamma)\\ &\le \max{}_{z\in\gamma}\frac{|f(z)|}{\epsilon^2\cdot\left(\frac{\epsilon}{2}\right)^2}\cdot\text{length}(\gamma).\end{aligned}$$

The bound is independent of h. Similarly for the bound of

$$\left|\oint_\gamma \frac{1}{(z-a)^3(z-a-h)^2}f(z)dz\right|.$$

$\square$

Suppose $f(z)$ is holomorphic at a, then it is differentiable on a disk centered at a of some radius $\epsilon > 0$. If $|z-a| < \epsilon$, then $f'(z)$ exists. Let γ be the circle centered at a with raidus $\frac{\epsilon}{2}$, so that $f(z)$ is dfferentiable inside γ. In fact, it's holomorphic inside γ. Since $f(z)$ is differentiable inside the small disk containing the given point, it's holomorphic at that point. So we can apply the Cauchy integral formula and its corollaries and conclude that $f''(a)$ exists. In particular, $f'(z)$ is differentiable at $z = a$. Doing this on a disk around a, we conclude $f'(z)$ is holomorphic at a.

Similarly $f'(z)$ being holomorphic at a implies that $f''(z)$ is holomorphic at a, which then implies $f'''(z)$ as well, and so on. So $f(z)$ being holomorphic implies f is infinitely differentiable.

7.4 Fundamental Theorem of Algebra

Theorem 7.4.1 (Fundamental Theorem of Algebra)**.** *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

is a polynomial with complex coefficients, then either the polynomial is constant or it can be written as a product of linear factors $(az + b)$*.*

Here's a weaker version of this theorem:

Theorem 7.4.2 (Fundamental Theorem of Algebra (Weaker version))**.** *If* $p(z)$ *is a non-constant polynomial, then there is some* w *such that* $p(w) = 0$ *(any non-constant polynomial has a root).*

Remark. If $az + b$ is a factor of $p(z)$, then $w = -b/a$ makes $aw + b = 0$, so $p(w) = 0$. If $p(w) = 0$, then $z - w$ divides $p(z)$. Then $\frac{p(z)}{z-w}$ is still a polynomial and we can repeat this until the polynomial becomes a constant.

Lemma 7.4.3. *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

is non-constant $(a_d \neq 0, d \geq 1)$*, then there is a real number* R *such that* $|z| \geq R$ *implies*

$$\frac{1}{2}|a_d z^d| \leq |p(z)| \leq \frac{3}{2}|a_d z^d|,$$

which is equivalent to

$$\frac{1}{2} \leq \left|\frac{p(z)}{a_d z^d}\right| \leq \frac{3}{2},$$

which is then equivalent to

$$\left|\left|\frac{p(z)}{a_d z^d}\right| - 1\right| \leq \frac{1}{2}.$$

Proof. We have

$$\begin{aligned}\frac{p(z)}{a_d z^d} - 1 &= \frac{a_d z^d}{a_d z^d} + \frac{a_{d-1} z^{d-1}}{a_d z^d} + \cdots + \frac{a_0}{a_d z^d} - 1 \\ &= \frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots \frac{a_0}{a_d} z^{-d}.\end{aligned}$$

Note when $|z| \to \infty, |z|^{-r} \to 0$ for any $r > 0$, i.e., for any $\epsilon > 0$, there is an R such that

$$|z| \geq R \implies \left||z|^{-r} - 0\right| < \epsilon.$$

We want R such that $|z| \geq R$ implies $|z|^{-r} < \epsilon$. Then

$$\begin{aligned}\log(|z|^{-r}) &< \log(\epsilon) \\ -r\log(|z|) &< \log(\epsilon) \\ \log(|z|) &> -\frac{\log(\epsilon)}{r} \\ |z| &> e^{-\frac{\log(\epsilon}{r}} = \epsilon^{-1/r}.\end{aligned}$$

We take any $R > \epsilon^{-1/r}$ to show that the limit exists.
Conclusion:

$$\frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots \frac{a_0}{a_d} z^{-d} \to 0$$

as $|z| \to \infty$, i.e., for any $\epsilon > 0$, there is an such that $|z| \geq R$ implies

$$\left|\frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots \frac{a_0}{a_d} z^{-d}\right| < \epsilon.$$

We take $\epsilon = \frac{1}{2}$ and conclude that for the resulting R, $|z| \geq R$ implies

$$\left|\frac{p(z)}{a_d z^d} - 1\right| < \epsilon = \frac{1}{2}.$$

□

Now we want to apply Liouville's theorem to $\frac{1}{p(z)}$. If $p(z) \neq 0$ for any z, then $\frac{1}{p(z)}$ is holomorphic (composition of $\frac{1}{z}$ with $p(z)$).

$$\begin{aligned}&\frac{1}{2}|a_d z^d| \leq |p(z)| \\ &\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_d|}|z|^{-d} \leq \frac{2}{|a_d|}R^{-d} \qquad \text{(since } |z| \geq R\text{)}\end{aligned}$$

To bound $\frac{1}{p(z)}$ on the disk of radius R centered at zero (i.e. $|z| \leq R$), notice that this region is compact (closed and bounded). Since $\frac{1}{p(z)}$ is holomorphic, it is continuous and $\left|\frac{1}{p(z)}\right|$ is the composition of the continuous functions $\frac{1}{p(z)}$ and absolute value, hence it is also continuous. Since the continuous image of a compact set is compact, the image of $\left|\frac{1}{p(z)}\right|$ on $|z| \leq R$ is compact (so in particular, it is bounded), i.e. $\left|\frac{1}{p(z)}\right| \leq M$ when $|z| \leq R$.

So now we combine the two bounds

$$\left|\frac{1}{p(z)}\right| \leq \max\left(\underbrace{\frac{2}{|a_d|}R^{-d}}_{\text{valid when } |z| \geq R}, \underbrace{M}_{\text{valid when } |z| \leq R}\right).$$

So for any choice of z, either $|z| \geq R$ or $|z| \leq R$, and the inequality holds. So we've shown $\frac{1}{p(z)}$ is a bounded holomorphic function. So by Liouville's theorem, it is constant.

Hence, if $p(z)$ is never zero, $\frac{1}{p(z)} = c$ is a constant, so $p(z) = \frac{1}{c}$. Since polynomial has no zero implies it is constant, we conclude that a polynomial being non-constant implies that the polynomial has a zero.

Remark. Not the only proof but a very typical application of Liouville's theorem (show some condition implies boundedness, deduce it's constant).

Chapter 8

Harmonic Functions

Recall the Cauchy-Riemann equations: if

$$f(x+iy) = u(x,y) + iv(x,y)$$

and if f is holomorphic, we have

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

8.1 Laplace Equation (2D)

We saw that if we can take a second derivative (and it's continuous, which guarantees $u_{xx} = u_{yx}$) we get

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}.$$

Then we obtain the $2D$ Laplace equation:

$$u_{xx} + u_{yy} = 0.$$

Recall that obeying C-R equations (and first derivatives being continuous) implies f is holomorphic, so it is infinitely differentiable $\implies$ it has second, third derivatives (second derivative differentiable $\implies$ it is continuous).

So actually, $f(x+iy)$ having continuous first derivatives obeying C-R equations is enough to deduce that $u(x,y), v(x,y)$ are solutions to the Laplace equation.

Definition 8.1.1 (Harmonic functions)**.** The solutions to the Laplace equation are called *harmonic functions*.

We only consider the $2D$ Laplace equation.

Goal: We want to figure out conditions under which a harmonic function is the real part of a holomorphic function.

For us, a solution of the Laplace equation is a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that the second derivatives $u_{xx}, u_{yy}, u_{xy}, u_{yx}$ exist and are continuous and we have $u_{xx} + u_{yy} = 0$ (this is a normal assumption when solving certain PDEs) (assuming continuity gives us $u_{xy} = u_{yx}$).

Theorem 8.1.2. *If $u(x, y)$ is a harmonic function on an open subset $G \subseteq \mathbb{R}^2$ with no "holes" (more precisely, for any closed curve $\gamma \in G$, the region enclosed by γ is also contained in G), then there is a harmonic function $v : G \to \mathbb{R}$ such that*

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic on G.

Remark. Such a $v(x, y)$ is called a *harmonic conjugate* of $u(x, y)$.

Idea: hard to write down $v(x, y)$ directly, but $f'(x + iy) = u_x(x, y) + iv_x(x, y)$ and if f is holomorphic, $v_x = -u_y$, so $f'(x+iy) = u_x(x, y) - iu_y(x, y)$. So $f'(x+iv)$ can be expressed in terms of u.

Proof. Define $g(x + iy) = u_x(x, y) - iu_y(x, y)$. Let's check it's holomorphic. Note that it has continuous first derivatives because u has continuous second derivatives, so it's enough that the C-R equations hold:

$$\begin{aligned}(u_x)_x &= u_{xx} = -u_{yy} = (-u_y)_y.\\ (u_y)_x &= u_{yx} = u_{xy} = -(-u_y)_x.\end{aligned}$$

Now we find an antiderivative of $g(z)$, i.e. a function $f(z)$ such that $f'(z) = g(z)$. We saw that this can be done when

(i) $g(z)$ is continuous,

(ii) $\oint_\gamma g(z)dz = 0$ for any closed curve $\gamma \in G$.

(The construction was to define $f(w) = \int_\delta g(z)dz$ where δ is any path from a fixed basepoint to w.)

Since $g(z)$ is holomorphic, it is differentiable and thus continuous. If γ is a closed curve in G, then by Cauchy's theorem

$$\oint_\gamma g(z)dz = 0$$

because $g(z)$ is holomorphic on G, in particular inside the region enclosed by γ. So such $f(z)$ exists and we write

$$f(x+iy) = a(x,y) + ib(x,y)$$

where $a, b : \mathbb{R}^2 \to \mathbb{R}$. Then $f'(z) = g(z)$ implies

$$a_x + ib_y = u_x - iu_y,$$

which means $a_x = u_x$ and $-a_y = b_x = -u_y \implies a_y = u_y$ by C-R equations applied to $f(z)$, which is holomorphic because it satisfies $f'(z) = g(z)$. Now let's integrate:

$$\begin{aligned} a_x = u_x &\implies a(x,y) = u(x,y) + C(y) \\ a_y = u_y &\implies a(x,y) = u(x,t) + D(x). \end{aligned}$$

Taking the difference we have

$$C(y) - D(x) = 0 \implies C(y) = D(x),$$

which implies C, D are constants that doesn't depend on x or y. So we have

$$a(x,y) = u(x,y) + c,$$

where c is some constant. Then

$$f(x+iy) - c = a(x,y) - c + ib(x,y) = u(x,y) + ib(x,y).$$

Thus, $f(z)$ is a holomorphic function whose real part is $u(x,y)$. □

If $G = \mathbb{C}\backslash\{0\}$, the theorem doesn't apply because unit circle encloses $0 \notin G$.

$$u(x,y) = \log(r) = \log(\sqrt{x^2+y^2}).$$

which is a harmonic function but it's not the real part of a holomorphic function on G. But if we replace G by $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ then it's the real part of $\log(z)$.

Corollary 8.1.3. *Any harmonic function is infinitely differentiable.*

Proof. If $u : G \to \mathbb{R}$ ($G \subseteq \mathbb{C}$ open), and $z \in G$ is the point where we want to check u is infinitely differentiable. Since G is open, it contains a open disk centered at z, radius $\epsilon > 0$. So it also contains the closed disk of radius $\epsilon/2$ centered at z. This small disk has no holes. Suppose $\gamma : [a,b] \to D(z, \epsilon/2)$ is a curve, then $|\gamma(t) - z| \leq \epsilon/2$. Any point in the interior of γ is still inside the disk. To see this, note that no point outside the disk can be enclosed by the curve γ. By looking at a small enough part of G near z, we can assume our function is the real part of a holomorphic function. Since holomorphic functions are infinitely complex differentiable, they are infinitely real differentiable and so are their real parts, so our function is infinitely differentiable. □

8.2 Maximum Modulus Principle

Proposition 8.2.1. *Suppose $u : G \to \mathbb{R}$ is a harmonic function where G is an open subset of $\mathbb{C}$. Suppose also that the closed disk of radius r centered at w is contained inside G. Then*

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})d\theta.$$

This is saying that $u(w)$ equals the average value of $u(z)$ on the circle.

Proof. We find $f(z)$ holomorphic such that $u(z) = \Re(f(z))$ for z in the closed disk, then we apply the Cauchy integral formula.

$$f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}dz$$

where γ is the circle centered at w with radius r, i.e., $\gamma(t) = w + re^{it}$ where $t \in [0, 2\pi]$.

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - w}\gamma'(t)dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f(\gamma(t))idt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(w + re^{it})dt. \end{aligned}$$

Then take real part to recover u. So if we have a harmonic function defined on a disk, the value at the center is the average of the values on the boundary. $\square$

Proposition 8.2.2. *Suppose u is a harmonic function on an open set containing the closed disk centered at w with radius r, and that $u(z) \leq u(w)$ for all $z \in G$. Then $u(z) = u(w)$ on the disk centered at w with radius r.*

Proof. If $u(w + re^{it}) = u(w)$ for all t, we have equality. In this case $u(z) = u(w)$ on the circle of radius r centered at w.

To recover the result for points inside the disk, apply the same argument with a smaller radius. So it's enough to show the result holds for z on the circle. We know $u(w + re^{it}) \leq u(w)$ by assumption. Suppose $u(w + re^{it}) \neq u(w)$ for some t_0. Then

$$u(w + re^{it_0}) < u(w).$$

Since u is harmonic, it is infinitely differentiable and thus continuous. So the $u(w + re^{it})$ is a continuous function of t because it is the composition of continuous functions. Hence, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|t - t_0| < \delta \implies |u(w + re^{it}) - u(w + re^{it_0})| < \epsilon.$$

Then

$$\begin{aligned} u(w+re^{it})-u(w+re^{it_0}) &\le |u(w+re^{it})-u(w+re^{it_0})| \le \epsilon = \frac{u(w)-u(w+re^{it_0})}{2} \\ 2u(w+re^{it})-2u(w+re^{it_0}) &\le u(w)-u(w+re^{it_0}) \\ u(w)-u(w+re^{it_0}) &\le 2u(w)-2u(w+re^{it}) \\ \epsilon &\le u(w)-u(w+re^{it}). \end{aligned}$$

Then

$$u(w+re^{it}) \le u(w)-\epsilon.$$

Define $g(t)=u(w)-u(w+re^{it})$. Then we have

$$\begin{aligned} \int_0^{2\pi} g(t)dt &\ge \int_{t_0-\delta}^{t_0+\delta} g(t)dt \\ &\ge \int_{t_0-\delta}^{t_0+\delta} \frac{g(t_0)}{2}dt = 2\delta\frac{g(t_0)}{2} > 0, \end{aligned}$$

which is a contradiction, so $g(t_0)=0$ as needed. □

Definition 8.2.3 (Path-connected)**.** A set G is *path-connected* if for any $p,q\in G$, there is a path with p,q as endpoints ($\gamma:[a,b]\to G$ where $\gamma(a)=p, \gamma(b)=q$).

Example 8.2.4. Any disk (open or closed) is path-connected because it is convex (if two points are in a convex set, the line segment joining them is also in the set, which is exactly the path we need). Therefore, any convex set is also path-connected.

Example 8.2.5 (Non-example)**.** The union of two disjoint open disks of radius 1 centered at 2 and -2 is not path-connected. Suppose $\gamma:[a,b]\to G$ with $\gamma(a)=-2$ and $\gamma(b)=2$. Consider $\Re(\gamma(t))$, a real-valued function which is continuous. Then by the Intermediate Value Theorem, there is $t\in[a,b]$ such that $\Re(\gamma(t))=0$. But there are not points in G with real part zero, which implies that $\gamma(t)\notin G$, a contradiction.

Example 8.2.6 (Cont'd)**.** Consider

$$g(p)=\begin{cases} 1 & \text{if } p \text{ is in right circle} \\ 0 & \text{if } p \text{ is in left circle.} \end{cases}$$

Note that $g(z)$ is holomorphic on G since

$$\lim_{h\to 0}\frac{g(p+h)-g(h)}{h}=0 \qquad\qquad (g(p+h)=g(h)).$$

$g(p)=\Re(g(p))$ so g may also be viewed as a harmonic function on G. The maximal value of g is 1 (attained on right disk) but g is not constant (takes different value on left disk).

We want to find a sequence of disks going along the curve joining w and z and conclude the function is constant on each disk, eventually covering the whole curve. We will fix a curve γ and then show that for any $\gamma(t)$, we can draw a disk of radius $\epsilon > 0$ (same ϵ for all $\gamma(t)$), so it will take roughly length$(\gamma)/\epsilon$ steps to go from one end to the other.

So we have $\gamma[a, b] \to G$ and we define $d : [a, b] \to \mathbb{R}_{\geq 0}$ such that

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z|.$$

$d(t)$ is the size of the largest disk we can draw at $\gamma(t)$. We want $d(t) \geq \epsilon > 0$ for some ϵ and all t. We will show

- $d(t) > 0$ for all t
- $d(t)$ is continuous.

Proposition 8.2.7. *If G is an open, path-connected region, then a harmonic function $u : G \to \mathbb{R}$ such that there is a $w \in G$ with $u(w) \geq u(z)$ for all $z \in G$ must be a constant function.*

Proof. We want to show $u(w) = u(z)$ by stepping along the path from w to z, showing u is constant along each step. If u is constant on a disk of radius ϵ centered at w, then we can go up to ϵ along the curve and u will still take the value $u(w)$.

The argument will be go $\epsilon/2$ (avoid reaching the boundary since disk is open) along the curve to a point ζ. We have $u(\zeta) = u(w)$, but the length of the part of γ from ζ to w is length$(\gamma) - \epsilon/2$. We can then repeat the argument to get a path of length length$(\gamma) - n\epsilon/2$ for n steps. Eventually this becomes $< \epsilon$, so z is inside the disk of radius ϵ centered at that point, which implies values are equal $(= u(w))$.

Conclusion: for any $z \in G$, $u(z) = u(w)$ so u is constant on G.

What we still need to check: we can use disks of the same radius ϵ at any point on γ. We need to find $\epsilon > 0$ and need the disk of radius ϵ centered at $\gamma(t)$ to be contained in G for all t. Equivalently, we want $\epsilon > 0$ such that

$$|\gamma(t) - z| < \epsilon \implies z \in G.$$

We have $G \subseteq \mathbb{C}$ open, $\gamma : [a, b] \to G$ is continuous. Define

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z|.$$

Note that $d(t) > 0$ because $\gamma(t) \in G$, and G is open, there is some disk centered at $\gamma(t)$ contained in G with radius $\epsilon(t)$, so any $z \in \mathbb{C} \setminus G$ is at least $\epsilon(t)$ away from $\gamma(t)$, i.e.

$|\gamma(t) - z| \geq \epsilon(t) > 0$, so $\epsilon(t)$ is a lower bound for the set $\{|\gamma(t) - z| : z \in \mathbb{C}\backslash G\}$ so it is less than or equal to the greatest lower bound, the infimum:

$$d(t) = \inf_{z \in \mathbb{C}\backslash G} |\gamma(t) - z| \geq \epsilon(t) > 0.$$

Now we show $d(t)$ is continuous, i.e.

$$\lim_{h \to 0} d(t+h) = d(t).$$

(since $d(t)$ is a distance (from $\gamma(t)$ to $\mathbb{C}\backslash G$), we will control it with the triangle inequality).

For $z \in \mathbb{C}\backslash G$, $|\gamma(t) - z| \geq d(t)$ by definition of $d(t)$, which implies $|\gamma(t+h) - z| \geq d(t+h)$. Then

$$\begin{aligned} d(t) \leq |\gamma(t) - z| &= |\gamma(t) - \gamma(t+h) + \gamma(t+h) - z| \\ &\leq |\gamma(t) - \gamma(t+h)| + |\gamma(t+h) - z|. \end{aligned}$$

So $d(t) - |\gamma(t) - \gamma(t+h)| \leq |\gamma(t+h) - z|$, which means $d(t) - |\gamma(t) - \gamma(t+h)|$ is a lower bound for $|\gamma(t+h) - z|$, which is less than the infimum. Hence,

$$d(t) - |\gamma(t) - \gamma(t+h)| \leq d(t+h).$$

Rewrite this as

$$d(t) - d(t+h) \leq |\gamma(t) - \gamma(t+h)|.$$

To get a lower bound, we use

$$\begin{aligned} d(t+h) \leq |\gamma(t+h) - z| &= |\gamma(t+h) - \gamma(t) + \gamma(t) - z| \\ &\leq |\gamma(t+h) - \gamma(t)| + |\gamma(t) - z|, \end{aligned}$$

so

$$d(t+h) - |\gamma(t+h) - \gamma(t)| \leq |\gamma(t) - z|.$$

Then $d(t+h) - |\gamma(t+h) - \gamma(t)| \leq d(t)$ and so

$$-|\gamma(t+h) - \gamma(t)| \leq d(t) - d(t+h).$$

Conclusion:

$$0 \leq |d(t+h) - d(t)| \leq |\gamma(t+h) - \gamma(t)|.$$

Now use squeeze theorem as $h \to 0$, by continuity of γ,

$$\lim_{h \to 0} \gamma(t+h) = \gamma(t).$$

So the RHS $\to 0$ and so $|d(t+h) - d(t)| \to 0$. Therefore $d(t+h) \to d(t)$, i.e. $d(t)$ is continuous. So we have $d : [a,b] \to \mathbb{R}$ (domain of $\gamma : [a,b] \to G$) with $d(t) > 0$ and d continuous. Since $[a,b]$ is compact, the set of values of $d(t)$ is again compact. By (Weierstrass theorem), a continuous function on a compact set has a minimum which it attains. Then the minimal value of $d(t)$ for $t \in [a,b]$ is some positive number ϵ (cannot be ≤ 0 because d never takes such values.)

Conclusion: $d(t) \geq \epsilon > 0$ for this ϵ. $\square$

Theorem 8.2.8 (Maximum modulus principle)**.** *Suppose $G \subseteq \mathbb{C}$ is open and path-connected, then $f : G \to \mathbb{C}$ is holomorphic if $|f(z)|$ attains a maximum in G, then $f(z)$ is constant.*

Chapter 9

Power Series

Definition 9.0.1 (Power series)**.**

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Example 9.0.2 (Taylor series)**.**

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + \cdots .$$

To understand this, we need to discuss what it means for a sequence of functions to converge.

Remark. The convergence of the sequence of partial sums $\sum_{n=0}^{k} a_n (z - z_0)^n$ is the same thing as convergence of the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

9.1 Convergence of sequences of functions

Suppose $G \subseteq \mathbb{C}$ and $f_n : G \to \mathbb{C}$ where $n \in \mathbb{Z}_{\geq 0}$ is a sequence of functions.

9.1.1 Pointwise convergence

Definition 9.1.1 (Pointwise convergence)**.** $f_n(z)$ converges to $f(z)$ *pointwise* if for each $z \in G$, the sequence $f_n(z)$ converges to $f(z)$, i.e. for each $z \in G$, for every $\epsilon > 0$, there is a $N \geq 0$ (can depend on z) such that $n \geq N$ implies

$$|f_n(z) - f(z)| < \epsilon.$$

Remark. Evaluating at z gives a sequence of complex numbers.

Example 9.1.2. Consider $f_n(z) = z^n$. $G = [0,1] \subseteq \mathbb{R}$. When $z = 1$, $f_n(z) = 1$, we get a constant sequence 1 and so $f_n(z) \to 1$. When $z < 1$, $f_n(z) \to 0$. We need to find N such that

$$\begin{aligned} |f_n(z) - 0| < \epsilon &\quad \text{for } n \geq N \\ z^n < \epsilon &\quad \text{for } n \geq N. \end{aligned}$$

This holds when

$$\begin{aligned} n\ln(z) &< \ln(\epsilon) \\ n &> \frac{\ln(\epsilon)}{\ln(z)} \qquad \text{(divide by } \ln(z) < 0 \text{ since } z \in [0,1]). \end{aligned}$$

Then we choose $N = \frac{\ln(\epsilon)}{\ln(z)}$.
Conclusion:

$$f_n(z) \to \begin{cases} 1 & z = 1, \\ 0 & 0 \leq z < 1. \end{cases}$$

which is not continuous.

To remedy this, we define a more stringent notion of convergence.

9.1.2 Uniform convergence

Definition 9.1.3 (Uniform convergence)**.** A sequence of functions $f_n(z)$ converges to $f(z)$ *uniformly* on G if for every $\epsilon > 0$, there exists $N \geq 0$ such that

$$|f_n(z) - f(z)| < \epsilon \quad \text{for all } n > N \text{ and } z \in G.$$

Remark. The difference from pointwise convergence is that N cannot depend on z.

Example 9.1.4 (Non-example)**.** Back to the $f_n(z) = z^n$ example. Now let $G = [0, r]$ where $r < 1$ is fixed.

$$\begin{aligned} |f_n(z) - 0| &< \epsilon \\ z^n &< \epsilon \\ n &> \frac{\ln(\epsilon)}{\ln(z)}. \end{aligned}$$

We need to find an N such that

$$N \geq \frac{\ln(\epsilon)}{\ln(z)} \quad \text{for all } z \in G = [0, r],$$

which is maximized at $z = r$ and so we can take $N = \frac{\ln(\epsilon)}{\ln(r)}$. Note though that while $f_n(z) \to 0$ on $G = [0, 1)$, it does not converge uniformly because we would need an N such that

$$N \geq \frac{\ln(\epsilon)}{\ln(z)} \quad \text{for all } z \in [0, 1).$$

But as z approaches 1, $\ln(\epsilon)/\ln(z) \to \infty$, so there is no such N for which all z obey the inequality.

Two main desirable properties of uniform convergence:

Proposition 9.1.5. *if $f_n(z)$ are continuous and converge uniformly to $f(z)$, then $f(z)$ is continuous.*

Proof. Fix z_0, we need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Let N be such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

for $n \geq N$ (use uniform convergence). Now pick some $n \geq N$ and let δ be such that

$$|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$$

whenever $|z - z_0| < \delta$. This follows from the continuity of $f_n(z)$. Finally, if $|z - z_0| < \delta$,

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \\ &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Thus, $f(z)$ is continuous. □

Proposition 9.1.6. *If $f_n(z)$ converges to $f(z)$ uniformly on G, and γ is a curve in G, then*

$$\lim_{n\to\infty} \oint_\gamma f_n(z)dz = \oint_\gamma f(z)dz.$$

(so we can swap the order if taking limits and integration).

Proof. For $\epsilon > 0$, pick N such that for all $n \geq N$ (use uniform convergence), we have

$$|f_n(z) - f(z)| < \frac{\epsilon}{\text{length}(\gamma)}.$$

$$\begin{aligned}\left|\oint_\gamma f_n(z)dz - \oint_\gamma f(z)dz\right| &= \left|\oint_\gamma f_n(z) - f(z)dz\right| \\ &\le \max_{z\in\gamma} |f_n(z) - f(z)| \cdot \text{length}(\gamma).\end{aligned}$$

But for all $z \in G$ (not just γ),

$$|f_n(z) - f(z)| < \frac{\epsilon}{\text{length}(\gamma)}.$$

Thus,

$$\begin{aligned}\left|\oint_\gamma f_n(z)dz - \oint_\gamma f(z)dz\right| &\le \frac{\epsilon}{\text{length}(\gamma)} \cdot \text{length}(\gamma) \\ &= \epsilon.\end{aligned}$$

□

9.1.3 Weierstrass M-test for uniform convergence

Theorem 9.1.7 (Weierstrass M-test)**.** *Suppose $G \subseteq \mathbb{C}, f_k : G \to \mathbb{C}$ and $M_k \in \mathbb{R}$ with $|f_k(z)| \le M_k$ for all $z \in G$. Then*

$$\sum_{k=0}^{\infty} M_k \textit{ converges} \implies \sum_{k=0}^{\infty} f_k(z) \textit{ converges uniformly on } G.$$

Proof. Suppose $\sum_{k=0}^{\infty} M_k$ converges to M. Since $0 \le |f_k(z)| \le M_k$, $M_k \ge 0$. For any $\epsilon > 0$, there exists $N > 0$ such that for $n \ge N$,

$$\left|\sum_{k=0}^{n} M_k - M\right| < \epsilon.$$

Then we have

$$\begin{aligned}\left|\sum_{k=0}^{n} M_k - \sum_{k=0}^{\infty} M_k\right| &< \epsilon \\ \left|-\sum_{k=n+1}^{\infty} M_k\right| &< \epsilon.\end{aligned}$$

Since each $M_k \ge 0$, we have

$$\sum_{k=n+1}^{\infty} M_k < \epsilon.$$

Now we need to show that
$$\left|\sum_{k=0}^{n} f_k(z) - \sum_{k=0}^{\infty} f_k(z)\right| < \epsilon.$$

Since
$$\left|\sum_{k=0}^{\infty} f_k(z)\right| \leq \sum_{k=0}^{\infty} |f_k(z)| \leq \sum_{k=0}^{\infty} M_k < \infty,$$

$\sum_{k=0}^{\infty} f_k(z)$ converges absolutely and so the limit of the sum exists.

$$\begin{aligned}\left|-\sum_{k=n+1}^{\infty} f_k(z)\right| &\leq \sum_{k=n+1}^{\infty} |f_k(z)| \\ &\leq \sum_{k=n+1}^{\infty} M_k \\ &< \epsilon.\end{aligned}$$

This no longer depends on z, so we just use the N from $\sum_k M_k$. □

Example 9.1.8. Let $f_k(z) = z^k$ where $|z| < 1$. Then
$$\sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Question. Does it converge uniformly?

Let's view $f_k(z)$ as functions on $|z| \leq r$ (here $0 \leq r < 1$ fixed). Convergence on this set is uniform.

Proof. Use M-test. We need $|f_k(z)| = |z^k| = |z|^k \leq r^k$. So choose $M_k = r^k$.
$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad (r < 1).$$

Conclusion: M-test applies and we have uniform convergence. □

Question. What about $|z| < 1$?

Answer. Convergence is not uniform. If it was, then for every $\epsilon > 0$, there is a $N > 0$ such that for $n \geq N$

$$\left|\sum_{k=0}^{n} z^k - \frac{1}{1-z}\right| < \epsilon$$
$$\left|\frac{1-z^{n+1}}{1-z} - \frac{1}{1-z}\right| < \epsilon$$
$$\frac{|z|^{n+1}}{|1-z|} < \epsilon.$$

However, the problem is that as $z \to 1$, this goes to infinity, so the bound $\frac{|z|^{n+1}}{|1-z|} < \epsilon$ cannot hold for all $|z| < 1$. Thus, the convergence is not uniform.

Remark. Even though $\{z : |z| < 1\}$ is the union of $\{z : |z| \leq r\}$ for $r < 1$, we don't have uniform convergence on the open unit disk.

9.2 Power Series

Lemma 9.2.1. *If $\sum_{k=0}^{\infty} a_k(w-z_0)^k$ converges, then $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely when $|z-z_0| < |w-z_0|$. Moreover, if $0 \leq \ell < |w-z_0|$, then convergence is uniform on $|z-z_0| \leq \ell$.*

Proof. Since $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges, the terms must go to zero (because if $\sum_{k=0}^{\infty} c_k$ converges, then we would have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \sum_{k=0}^{n} c_k - \sum_{k=0}^{n-1} c_k = 0.)$$

Therefore,

$$\lim_{k\to\infty} a_k(w-z_0)^k = 0,$$

i.e., for any $\epsilon > 0$, there is $N > 0$ such that for $n \geq N$,

$$|a_n(w-z_0)^n - 0| < \epsilon.$$

Thus, $|a_n(w-z_0)^n| < \epsilon$. Now let

$$M = \max\{|a_0(w-z_0)^0|, \ldots, |a_N(w-z_0)^N|, \epsilon\},$$

so $M \geq |a_k(w-z_0)^k|$ for all k. Then

$$\sum_{k=0}^{\infty} |a_k(z-z_0)^k| = \sum_{k=0}^{\infty} |a_k(w-z_0)^k| \left|\frac{z-z_0}{w-z_0}\right|^k$$
$$\leq \sum_{k=0}^{\infty} M \left|\frac{z-z_0}{w-z_0}\right|^k.$$

Since this is a geometric series with common ratio $\left|\frac{z-z_0}{w-z_0}\right| < 1$, it converges.
Conclusion: we have absolute convergence.

For uniform convergence, use M-test:

$$|a_k(z-z_0)^k| \le M\left|\frac{z-z_0}{w-z_0}\right|^k = M_k \le \left|\frac{\ell}{w-z_0}\right|^k \qquad (w-z_0<1).$$

We need sum of M_k to converge:

$$\sum_k M_k = \sum_k M\left(\frac{\ell}{|w-z_0|}\right)^k,$$

which is a geometric series with common ratio < 1, so it converges and is uniformly convergent by M-test. $\square$

Theorem 9.2.2. *For a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, there is a $R \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ such that the series converges absolutely when $|z-z_0| < R$ and diverges when $|z-z_0| > R$. It converges uniformly on $|z-z_0| \le \ell$ ($\ell < R$ is fixed).*

Proof. Consider

$$S = \left\{ x \in \mathbb{R}_{\ge 0} \;\mid \sum_{k=0}^{\infty} a_k x^k \text{ converges} \right\}.$$

Note that $0 \in S$, and so S is nonempty (can ask for supremum, which will be in $\mathbb{R}_{\ge 0} \cup \{\infty\}$. Applying the lemma above, if $x \in S$, then $\sum_{k\ge 0} a_k x^k$ converges and so $\sum_{k\ge 0} a_k(z-z_0)^k$ converges absolutely for $|z-z_0| < |x| = x$ ($w = z_0 + x$). So if we let $z = z_0 + y$, then

$$|y| < x \implies \sum_{k\ge 0} a_k y^k \quad \text{converges absolutely} \qquad \implies y \in S.$$

y can be in $[0,x)$. So $x \in S \implies [0,x) \subseteq S \implies [0,x] \subseteq S$. Consider the following cases:

1. $\sup S = \infty$ (S is unbounded): S contains a sequence x_i with limit ∞. Then S contains $[0, x_i]$ for all i. For any real number t, eventually $x_i \ge t$. Then $t \in [0, x_i] \subseteq S$. Hence, $t \in S$ and $S = \mathbb{R}_{\ge 0}$ ($R = \infty$).

2. $\sup S = R$. If $|z-z_0| > R$ and $\sum_{k\ge 0} a_k(z-z_0)^k$ converges, then S contains all x with $0 \le x < |z-z_0|$. But R is an upper bound for S so it must be at least as big as any such x, for example,
$$x = \frac{R + |z-z_0|}{2} > R.$$
This contradicts R being an upper bound for S. So $|z-z_0| > R \implies$ divergence. Now if $|z-z_0| < R$, then S contains an element x such that $|z-z_0| < x \le R$ (otherwise

$|z-z_0|$ would be an upper bound for S strictly less than R, the least upper bound). Now $\sum_{k\geq 0} a_k x^k$ converges as $x \in S$ and $|z-z_0| < x \implies$ absolute convergence for $|z-z_0| < x$ (by lemma). Hence, we have absolute convergence for $|z-z_0| < R$.

$\square$

Definition 9.2.3 (Radius of convergence)**.** R is the *radius of convergence* of $\sum_{k\geq 0} a_k(z-z_0)^k$ where $|z-z_0| < R$ (region on which we have convergence).

Remark. We cannot conclude anything about convergence when $|z-z_0| = R$.

Lemma 9.2.4. *The radius of convergence R obeys*

$$\frac{1}{R} = \lim sup \sqrt[k]{|a_k|}.$$

Proof. If $\limsup \sqrt[k]{|a_k|} = L$. For every $\epsilon > 0$, there exists N such that $k \geq N$ implies

$$\sqrt[k]{|a_k|} > L + \epsilon.$$

Then

$$|a_k| > (L+\epsilon)^k.$$

Hence,

$$\begin{aligned}\sum_{k\geq 0} |a_k(z-z_0)|^k &\leq \sum_{k=0}^{N} |a_k||z-z_0|^k + \sum_{k>N} |a_k||z-z_0|^k \\ &= \text{constant} + \sum_{k>N} (L+\epsilon)^k |z-z_0|^k.\end{aligned}$$

This converges when $(L+\epsilon)|z-z_0| < 1$ for some $\epsilon > 0$, equivalently,

$$|z-z_0| < \frac{1}{L+\epsilon},$$

and so we have absolute convergence when $|z-z_0| < 1/L$. We conclude $R \geq 1/L$ because if $1/L > R$, choose z such that $1/L > |z-z_0| > R$ (first inequality implies absolute convergence but the second inequality implies divergence, contradiction!)

If $L = \limsup \sqrt[k]{|a_k|}$, for any $\epsilon > 0$, there are infinitely many k such that $\sqrt[k]{|a_k|} > L - \epsilon$. Assume $L > 0$ (need to show $L = 0 \iff R = \infty$) for ϵ small enough such that $L - \epsilon > 0$. For infinitely many k, we have

$$\begin{gathered}|a_k| > (L-\epsilon)^k \\ |a_k||z-z_0|^k \geq (L-\epsilon)^k |z-z_0|^k.\end{gathered}$$

If converged, it would go to zero and then $(L-\epsilon)|z-z_0| < 1$. If $\frac{1}{L-\epsilon} \leq |z-z_0|$, we have divergence. So $\frac{1}{L} < |z-z_0|$ implies divergence.

Claim.

$$\frac{1}{L} \geq R$$

Proof. If not, we have a contradiction because

$$R > |z - z_0| > \frac{1}{L}$$

implies both convergence and divergence. $\square$

Since we have $\frac{1}{L} \geq R$ and $\frac{1}{L} \leq R$, we have $\frac{1}{L} = R$. $\square$

Remark. If $L = 0$, any $\epsilon > 0$ is an upper bound on all but finitely many terms $\sqrt[k]{|a_k|}$.

Example 9.2.5. Consider $a_k = \frac{1}{k!}, z_0 = 0, \sum_{k\geq 0} \frac{z^k}{k!} = e^z$. Then we can find the radius of convergence via

$$\frac{1}{R} = \limsup \sqrt[k]{\left|\frac{1}{k!}\right|}.$$

To compute this directly, we can use stirling's formula:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

We can see that

$$\sqrt[n]{n!} \approx \frac{n}{e} + \text{ lower terms.}$$

Therefore,

$$\sqrt[n]{\frac{1}{n!}} \approx \frac{e}{n} \to 0 \quad \text{as } n \to \infty.$$

So $R = \infty$. We can also use ratio test.

Example 9.2.6. $\sum_k a_k(z - z_0)^k = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$ where $z_0 = 0$ and

$$a_k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k-1)/2}}{k} & k \text{ odd.} \end{cases}$$

$$\limsup_{k \text{ odd}} \sqrt[k]{\left|\frac{(-1)^{(k-1)/2}}{k}\right|} = \sqrt[k]{\frac{1}{k}}.$$

Taking log gives

$$\ln\left(\sqrt[k]{\frac{1}{k}}\right) = \frac{1}{k}(-\ln(k)) \to 0.$$

Then

$$\sqrt[k]{\frac{1}{k}} \to 1.$$

Proposition 9.2.7. *If $\sum a_k(z-z_0)^k$ has radius of convergence R, then the limit function is continuous on the open disk centered at z_0 with radius at R.*

Proof. We want to argue that uniform convergence of continuous things gives something continuous. Let z be in the disk $|z-z_0| < R$. z is contained in the disk $|w-z_0| \leq \frac{|z-z_0|+R}{2}$.By what we know, convergence is uniform here, hence it is continuous on this disk, in particular at z. So for any z in the disk, we have continuity (i.e., the power series is continuous). $\square$

Proposition 9.2.8. *If γ is a curve in the open disk $|z-z_0| < R$ centered at z_0, then*

$$\oint_\gamma \sum_{k\geq 0} a_k(z-z_0)^k dz = \sum_{k\geq 0} a_k \oint_\gamma (z-z_0)^k dz.$$

If γ is a closed curve, then this is zero.

Proof. If we have uniform convergence, then

$$\begin{aligned}\oint_\gamma \sum_{k\geq 0} a_k(z-z_0)^k dz &= \lim_{n\to\infty} \oint_\gamma \sum_{k=0}^{n} a_k(z-z_0)^k dz \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \oint_\gamma a_k(z-z_0)^k dz \\ &= \sum_{k=0}^{\infty} \oint_\gamma a_k(z-z_0)^k dz.\end{aligned}$$

So it would be enough for the series to converge uniformly on a set containing the curve γ. We want to find a disk of radius $< R$ still containing γ. When we discussed how harmonic functions, we considered

$$d(t) = \inf_{z\in\mathbb{C}\setminus G} |\gamma(t) - z|.$$

We showed $d(t) \geq \epsilon > 0$.

Claim. The disk of radius $R - \epsilon/2$ contains γ.

If there was a point on γ outside this region, then it is within $\epsilon/2$ of a point outside the disk of radius R. This would contradict

$$\inf |\gamma(t) - z| \geq \epsilon.$$

We would have produced a value of this that was at most $\epsilon/2$. So we can find such a disk and we have uniform convergence.

If γ is closed, by Cauchy's theorem

$$\oint_\gamma (z - z_0)^k = 0.$$

Then

$$\oint_\gamma \sum_{k\geq 0} a_k(z - z_0)^k dz = \sum_{k=0}^{\infty} a_k \cdot 0 = 0.$$

□

Theorem 9.2.9 (Morera's Theorem)**.** *Suppose $f(z)$ is continuous on an open set G and for any closed curve γ in G,*

$$\oint_\gamma f(z)dz = 0.$$

Then $f(z)$ is holomorphic.

Proof. Recall that if $f(z)$ is continuous and $\oint_\gamma f(z)dz = 0$ for any closed curve γ, then it has an antiderivative $F(z)$ such that $F'(z) = f(z)$. Note that $F(z)$ is differentiable on G, so actually holomorphic on G. But holomorphic functions are infinitely differentiable. So F is twice differentiable, and thus $F'(z) = f(z)$ is differentiable on G. Thus, $f(z)$ is holomorphic on G. □

Theorem 9.2.10. *If $\sum_{k\geq 0} a_k(z - z_0)^k$ has radius of convergence R, then on $|z - z_0| < R$, it defines a holomorphic function.*

Proof. We saw that it's continuous, and the integrals along closed curves are zero, so Morera's theorem applies. □

Now let's understand derivatives of power series.

Lemma 9.2.11. *If $f(z) = \sum_{k\geq 0} a_k(z - z_0)^k$ for $|z - z_0| < R$, then for such z,*

$$f'(z) = \sum_{k\geq 0} k a_k (z - z_0)^{k-1}.$$

Proof. Recall that

$$f'(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z)^2} dw,$$

where γ is a closed curve that encloses z. Then

$$f'(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\sum_{k\geq 0} a_k(w - z_0)^k}{(w - z)^2} dw.$$

We want this series to converge uniformly on a set containing γ (actually we consider the set γ). Note that $\gamma(t)$ is continuous, and so $\left|\frac{1}{(\gamma(t)-z)^2}\right|$ is continuous as z is not on the curve γ. Therefore, $\left|\frac{1}{(\gamma(t)-z)^2}\right|$ is a continuous function $f : [a,b] \to \mathbb{R}$. Then its values are bounded since $[a,b]$ is compact.

Claim. If the series $\sum_{n\geq 0} g_n(z)$ converges uniformly to $g(z)$ and $h(z)$ is bounded, then

$$\sum_{n\geq 0} g_n(z)h(z) \to g(z)h(z) \quad \text{uniformly.}$$

Proof. Given

$$\left|\sum_{n=0}^{k} g_n(z) - g(z)\right| < \epsilon$$

for any $\epsilon > 0$, there is N such that $k \geq N$ implies this holds for all z.

$$\begin{aligned}\left|\sum_{n=0}^{k} g_n(z)h(z) - g(z)h(z)\right| &= \left|\sum_{n=0}^{k} g_n(z) - g(z)\right| |h(z)| \\ &\leq \epsilon \cdot \sup_z |h(z)|.\end{aligned}$$

This holds uniformly. □

Now back to the original proof, we have $g_n(w) = a_n(w-z_0)^n$ and $h(w) = \frac{1}{(w-z)^2}$. Then

$$\begin{aligned}f'(z) &= \frac{1}{2\pi i}\sum_{k\geq 0} a_k \oint_\gamma \frac{(w-z_0)^k}{(w-z)^2} dw \\ &= \frac{1}{2\pi i}\sum_{k\geq 0} a_k 2\pi i \left.\frac{\partial}{\partial w}(w-z_0)^k\right|_{w=z} \\ &= \sum_{k\geq 0} a_k k(z-z_0)^{k-1}.\end{aligned}$$

If $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$, then

$$\begin{aligned}f(z_0) &= a_0 \\ f'(z_0) &= 1 \cdot a_1 \\ f''(z_0) &= 2 \cdot \cdot a_2 \\ f^{(n)}(z_0) &= k! a_k.\end{aligned}$$

Thus,

$$f(z) = \sum_{k\geq 0} a_k(z-z_0)^k = \sum_{k\geq 0} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k.$$

$\square$

Example 9.2.12. Consider $z_0 = 0$ and $a_k = 1/k!$. Then

$$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

We will show that this is equal to e^z. Note that

$$f'(z) = \sum_{k=0}^{\infty} \frac{k}{k!}(z-z_0)^{k-1} = f(z).$$

$f(0) = 1$. Now consider $f(z)e^{-z}$. Then taking the derivative gives

$$f'(z)e^{-z} - f(z)(e^{-z}) = 0.$$

This implies that $f(z)e^{-z} = C$ is a constant. At $z = 0$, we have $f(0)e^{-0} = C \implies C = 1$. Thus

$$f(z)e^{-z} = 1 \implies f(z) = e^z.$$

Theorem 9.2.13. *Suppose $f(z)$ is holomorphic on $|z-z_0| < R$. Then*

$$f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$$

where

$$a_k = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw$$

and γ is a curve enclosing z_0 but contained in the disk. The series converges on $|z-z_0| < R$.

Proof. By Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}dw.$$

Now write

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}.$$

We want to expand the last factor as a geometric series to make sure $|w-z_0| > |z-z_0|$ on γ. We choose γ to be the circle $|w-z_0| = \frac{|z-z_0|+R}{2}$. Then we have uniform convergence on the enclosed region and so

$$\sum_{k\geq 0} \left(\frac{z-z_0}{w-z_0}\right)^k \xrightarrow{\text{uniformly}} \frac{1}{1-\frac{z-z_0}{w-z_0}}.$$

Consider

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z_0}\cdot \sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k dw &= \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}dw \\ &= f(z).\end{aligned}$$

We appealed to uniform convergence to interchange the order of integration and summation. Although $\frac{f(w)}{w-z_0}$ is not defined at $w = z_0$, it is defined and bounded on γ (because it is continuous, and γ is a compact set). So we can write

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\oint_\gamma \frac{f(w)}{w-z_0}\frac{(z-z_0)^k}{(w-z_0)^k}dw \\ &= \sum_{k=0}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z_0)^{k+1}}dw\right]}_{a_k}(z-z_0)^k.\end{aligned}$$

Notice that

$$\oint_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw$$

is the same for any γ enclosing z_0. The argument with uniform convergence implies that this series converges for our particular choice of z. We could do this for any z with $|z-z_0| < R$, so we have convergence for such z. Hence, the radius of convergence is at least R (but possibly could be larger). □

Theorem 9.2.14. *The radius of convergence equals to the largest number R' such that $f(z)$ can be extended to a holomorphic function on $|z-z_0| < R'$.*

Proof. Previous result tells us that the radius of convergence R is at least R' ($R \geq R'$). On the other hand, if the radius of convergence is R, the power series defines a holomorphic function extending $f(z)$ on $|z-z_0| < R$. So $R' \geq R$. Hence, $R = R'$. □

Example 9.2.15. Consider

$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots = \arctan(z).$$

Previously we saw that the radius of convergence is 1. Even though $\arctan(x)$ is infinitely differentiable as a function of a real variable, the Taylor series still only has finite radius of convergence (as opposed to e^x). Actually $\arctan(z)$ does not extend holomorphically to $z = \pm i$. If $\tan(z) = i$, then $\sin(z) = i\cos(z)$ and so $\sin^2(z) + \cos^2(z) = 0$. But $\sin^2(z) + \cos^2(z) = 1$ for any z, so there is no such z. Therefore, we cannot extend $\arctan(z)$ outside $|z| < 1$ holomorphically. Thus, $R = 1$ as we already checked.

Alternatively,

$$\arctan(z) = \int_0^z \frac{1}{1+w^2}dw,$$

which blows up at $w = \pm i$.

Proposition 9.2.16.

$$\frac{\partial^k f}{\partial z^k}(z_0) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw$$

where γ encloses z_0. (We already saw this for $k = 0, 1, 2$)

Proof. Our power series is a Taylor series:

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

where

$$a_k = \frac{1}{k!}\frac{\partial^k f}{\partial z^k}(z_0).$$

We get that by differentiating k times and evaluating at z_0. Then compare this to the formula for a_k from the theorem. □

Remark. Functions that can be written as a convergent power series on a disk centered at z_0 are called *analytic* (at z_0). In complex analysis, holomorphic and analytic mean the same thing.

Recall that if $p(z)$ is a non-constant polynomial and if $p(z_0) = 0$, then

$$p(z) = (z-z_0)\cdot\frac{p(z)}{z-z_0},$$

where $\frac{p(z)}{z-z_0}$ is still a polynomial. We can do the same thing for power series and holomorphic functions.

Theorem 9.2.17. *If $f(z)$ is holomorphic at z_0 and $f(z_0)$, then either $f(z)$ is constant (on some neighborhood of z_0) or there is $m \in \mathbb{Z}_{\geq 0}$ such that*

$$f(z) = g(z)(z - z_0)^m$$

and

(i) $g(z)$ is holomorphic at z_0.

(ii) $g(z_0) \neq 0$.

Remark. This m is called the order of the zero z_0 for $f(z)$ or the order of vanishing of $f(z)$ at z_0.

Proof. If

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k,$$

either all $a_k = 0$, in which case $f(z) = \sum_{k \geq 0} 0 = 0$ (constant), or there is m such that $a_0, a_1, \ldots, a_{m-1}$ are zero and $a_m \neq 0$, then

$$f(z) = (z - z_0)^m \sum_{k \geq m} a_k (z - z_0)^{k-m}.$$

or equivalently,

$$f(z) = (z - z_0)^m \underbrace{\sum_{k \geq 0} a_{k+m} (z - z_0)^k}_{g(z)}.$$

We check that $g(z)$ is holomoprhic at z_0 and $g(z_0) \neq 0$. Since convergent power series are holomorphic, it suffice to show that this has positive radius of convergence.

$$\frac{1}{R} = \limsup \sqrt[k]{|a_{k+m}|}.$$

Since the a_k came from the power series

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k,$$

which is holomorphic at z_0, then this implies that it has a positive radius of convergence. Hence, we have

$$\limsup \sqrt[k]{|a_k|} < \alpha \quad \text{for some } \alpha > 0.$$

So for all k large enough $\sqrt[k]{|a_k|} < \alpha$ and hence $|a_k| < \alpha^k$ and $|a_{k+m}| < \alpha^{k+m}$, so

$$\sqrt[k]{|a_{k+m}|} < \alpha^{\frac{k+m}{k}} = \alpha^{1+\frac{m}{k}} \leq \alpha^{1+m},$$

so

$$\limsup \sqrt[k]{|a_{k+m}|} \leq \alpha^{1+m}.$$

Then we have

$$g(z) = \sum_{k \geq 0} a_{k+m}(z - z_0)^k$$

converges on some disk centered at z_0, which defines a holomorphic function at z_0.

$$g(z_0) = a_m \neq 0.$$

Note that if there is no a_m that is non-zero, then

$$f(z) = \sum_{k \geq 0} a_k(z - z_0)^k = \sum_{k \geq 0} 0 = 0.$$

Therefore, f is the constant function equal to zero. □

9.3 Identity Principle

Corollary 9.3.1. *If $f(z)$ is holomorphic at z_0 and $f(z_0) = 0$, then there is an open disk centered at z_0 on which f takes the value 0 only at z_0.*

Proof. Write $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$ and $g(z) \neq 0$ is continuous at z_0 (since it's holomorphic). Then for every $\epsilon > 0$, there is $\delta > 0$ such that if $|z - z_0| < \delta$, then $|g(z) - g(z_0)| < \epsilon$. Take $\epsilon = |a_m|$, then

$$|g(z) - a_m| < |a_m|,$$

which forbids $g(z) = 0$ for $|z - z_0| < \delta$. So zeros of holomorphic functions are isolated: each is contained in an open set containing no other points. □

Theorem 9.3.2 (Identity principle)**.** *If $f(z)$ is holmorphic on an open set G, and $z_1, z_2, z_3, \ldots$ are points in G with $f(z_i) = 0$, if (z_i) has an accumulation point in G, then $f(z)$ is the zero function.*

Proof. Let w be an accumulation point of z_j in G, so there is a subsequence z_{i_j} converging to w. Since f is holomorphic in G, it is continuous at w.

$$f(w) = f\left(\lim_j z_{i_j}\right) = \lim_j f(z_{i_j}) = \lim_j 0 = 0.$$

Now this contradicts the previous result because $f(z)$ must be nonzero on some disk centered at w (except at w itself) if it is nonconstant. But the subsequence z_{i_j} must get arbitrarily close to w, so it would intersect any disk centered at w, and give a point where f vanishes. Therefore, the function cannot be nonconstant and so we must have $f(z) = 0$. □

Corollary 9.3.3. *If $f(z)$, $g(z)$ are holomorphic on G, and agree on a set $\{z_1, z_2, \ldots\}$ with an accumulation point ($f(z_i) = g(z_i$ for each i), then $f(z) = g(z)$.*

Proof. Apply the previous theorem to $f(z) - g(z)$ (holomorphic on G, vanishes at z_i), then $f(z) - g(z) = 0$. Thus, $f(z) = g(z)$. □

Remark. Writing

$$f(z) = \sum_k a_k (z - z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0)^{k+1}} dw.$$

Then $f(z)$ is determined by its power series, hence by a_k, which is determined by values of $f(z)$ along γ, a curve around z_0.

Conclusion: $f(z)$ is determined by its behavior on an open set containing z_0.

We have shown that $f(z)$ is determined by its value on a set having z_0 as an accumulation point. This could be a discrete set. Although we know that if $g(z)$ is another holomorphic function agreeing with $f(z)$ on this set then $f(z) = g(z)$ (so the set of values determines f). It isn't clear how to find values of $f(z)$ at other points from this information.

Example 9.3.4. $\sin\left(\frac{1}{z}\right)$ holomorphic for $z \neq 0$ (composition of holomorphic functions). Then the zeros are $z_n = \frac{1}{\pi n}$ for $n = 1, 2, \ldots$. z_n converges to 0. But $\sin\left(\frac{1}{z}\right)$ is not constant. Not a contradiction because the accumulation point is outside the set G where the function is holomorphic.

Chapter 10

Laurent Series

Definition 10.0.1 (Laurent Series)**.** A *Laurent series* is a series of the form

$$\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k.$$

Here

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

converges when $|z-z_0| < R_2$ and

$$\sum_{k=-\infty}^{-1} a_k(z-z_0)^k = \sum_{k=1}^{\infty} a_{-k}((z-z_0)^{-1})^k$$

is a power series in the variable $(z-z_0)^{-1}$ that converges when $|(z-z_0)^{-1}| < \frac{1}{R_1}$, i.e. $R_1 < |z-z_0|$. Thus, the Laurent series converges when the points satisfies $R_1 < |z-z_0| < R_2$, i.e., they lie in an annulus (flat doughnut shape).

Remark. Sometimes $R_1 \geq R_2$, in which case the series might never converge. For example, consider $a_k = 1$ for all k. Then we have

$$\sum_{k=-\infty}^{\infty} z^k = \sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} z^k.$$

The first summation from the RHS converges only when $|z^{-1}| < 1$ and the second summation converges only when $|z| < 1$. However, we cannot have both $|z| < 1$ and $|z^{-1}| < 1$. Hence, it doesn't converge for any z.

Theorem 10.0.2. *Suppose $f(z)$ is holomorphic in the annulus $R_1 < |z - z_0| < R_2$. Then*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

with

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0)^{k+1}} dw$$

(γ is any curve inside the annulus enclosing z_0, usually a circle centered at z_0 with radius between R_1 and R_1).

Proof. Modify the argument for power series. By Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma'} \frac{f(w)}{w - z} dw$$

where γ' is a curve around z. inside of which $f(z)$ is holomorphic. We choose γ' to be

- anticlockwise big circle (encloses z, smaller radius than R_2),
- clockwise small circle (doesn't enclose z, radius larger than R_1).

Then

$$f(z) = \frac{1}{2\pi i} \left[\int_{\text{big circle}} \frac{f(w)}{w - z} dw - \int_{\text{small circle}} \frac{f(w)}{w - z} dw \right].$$

Here the minus sign accounts for the orientation. (Note that if f is holomorphic for all $|z - z_0| < R_2$, then the second integral is zero by Cauchy's theorem, and we are in the same situation as when we discussed power series.) Instead of rewriting $\frac{1}{w_z}$ as follows

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{1}{w - z_0} \sum_{k \geq 0} \left(\frac{z - z_0}{w - z_0} \right)^k,$$

we write

$$\frac{1}{w - z} = \frac{-1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}} = \frac{-1}{z - z_0} \sum_{k \geq 0} \left(\frac{w - z_0}{z - z_0} \right)^k,$$

which is a geometric series if $|w - z_0| < |z - z_0|$. Then apply the uniform convergence of geometric series to interchange order of summation and integration. Then we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\text{big circle}} \frac{f(w)}{w - z} dw &= \frac{1}{2\pi i} \int \frac{f(w)}{w - z_0} \sum_{k \geq 0} \left(\frac{z - z_0}{w - z_0} \right)^k dw \\ &= \sum_{k \geq 0} \frac{1}{2\pi i} \int_{\text{big circle}} \frac{f(w)}{(w - z_0)^{k+1}} dw (z - z_0)^k. \end{aligned}$$

$$\frac{-1}{2\pi i}\int_{\text{small circle}}\frac{f(w)}{w-z}dw = \frac{-1}{2\pi i}\int\frac{-f(w)}{z-z_0}\sum_{k\geq 0}\left(\frac{w-z_0}{z-z_0}\right)^k dw$$
$$= \sum_{k\geq 0}\frac{1}{2\pi i}\int_{\text{small circle}}\frac{f(w)(w-z_0)^k}{(z-z_0)^{k+1}}dw.$$

□

Conclusion: $f(z)$ determines the coefficients of its Laurent series so there is an unique Laurent series in a given annulus.

Remark. Note though that a given function can have different Laurent series in different annuli. For example, consider $f(z) = \frac{1}{1-z}$ which have singularity at $z = 1$. If $0 < |z| < 1$, then

$$f(z) = \sum_{k=0}^{\infty} z^k$$

is a geometric series (Laurent series is just a power seris). If $1 < |z| < \infty$,

$$f(z) = \frac{-1}{z}\cdot\frac{1}{1-z^{-1}} = \frac{-1}{z}\sum_{k\geq 0}(z^{-1})^k = -\sum_{k=1}^{\infty} z^{-k}.$$

Recall

$$a_k = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw.$$

Note that

$$2\pi i a_{-1} = \oint_\gamma f(w)dw.$$

So if we can find a_{-1}, we can find integrals of $f(w)$. Also note that by Cauchy's theorem, we have

$$\oint_\gamma f(z)dz = 0.$$

Hence,

$$2\pi i a_{-1} = 0 \implies a_{-1} = 0.$$

Now consider the outer region where γ is a anticlockwise circle with radius greater than 1. Then

$$\oint_\gamma f(z)dz = \oint_\gamma \frac{-1}{z-1}dz = 2\pi i(-1) = 2\pi i a_{-1}.$$

So we have $a_{-1} = -1$.

Viewing a Laurent series as

$$\underbrace{\sum_{k=0}^{\infty} a_k(z-z_0)^k}_{\text{converges for } |z-z_0| < \text{distance to some singularity}} \quad + \quad \underbrace{\sum_{k=-1}^{-\infty} a_k(z-z_0)^k}_{\text{converges for } \left|\frac{1}{z-z_0}\right| < \text{distance to some singularity}},$$

we find that we can't have convergence outside the annulus.

Example 10.0.3. Consider

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

Then consider three regions $0 < |z| < 1$, $1 < |z| < 2$ and $2 < |z| < \infty$. First notice that

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}.$$

If $|z| > 2$,

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{z}{2}} = \frac{1}{z}\sum_{k\geq 0}\left(\frac{2}{z}\right)^k.$$

If $|z| < 2$,

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-\frac{z}{2}} = \frac{-1}{2}\sum_{k\geq 0}\left(\frac{z}{2}\right)^k.$$

Similarly there are two expansions for $\frac{1}{z-1}$. Add up the series for each term which are valid in the annulus we chose to work in.

10.1 Isolated Singularities

Recall that the identity principle states that if the set of zeros of a holomorphic function has an accumulation point (inside the domain) then the function is the zero function.

Conclusion 1: the zeros of a non-constant holomorphic function are isolated: for any z_0 with $f(z_0) = 0$, there is $\epsilon > 0$ such that $f(z) \neq 0$ for $0 < |z - z_0| < \epsilon$.

Definition 10.1.1 (Isolated)**.** A singularity z_0 of a function $f(z)$ is *isolated* if there is $\epsilon > 0$ such that $f(z)$ is holomorphic on $0 < |z - z_0| < \epsilon$.

Remark. This is an annulus: there is a Laurent series.

Definition 10.1.2. If z_0 is an isolated singularity of $f(z)$, then

(a) if $f(z)$ extends to a holomorphic function on $|z - z_0| < \epsilon$, we say z_0 is a *removable singularity*;

(b) if $\lim_{z\to z_0} |f(z)| = \infty$, then $f(z)$ has a *pole* at z_0;

(c) otherwise z_0 is called an *essential singularity.*

Example 10.1.3. $f(z) = \frac{e^z-1}{z}$ at $z = 0$. Recall if $f(z_0) = 0$, we can write

$$f(z) = (z - z_0)^m g(z)$$

for some $g(z)$ holomorphic at z_0 and $g(z_0) \neq 0$. Let $f(z) = e^z - 1$, $f(0) = 0$, so $f(z) = z^m g(z)$. Turns out $m = 1$,

$$g(z) = \begin{cases} \frac{e^z-1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}.$$

Example 10.1.4. $f(z) = \frac{1}{z^2}$ at $z = 0$. Then

$$\lim_{z\to 0} \left| \frac{1}{z^2} \right| = \infty.$$

Similarly for $\frac{1}{z}$, $\frac{1}{z^7}$, etc..

Example 10.1.5. $f(z) = e^{\frac{1}{z}}$.

$$\lim_{z\to 0^+} e^{\frac{1}{z}} = \infty$$
$$\lim_{z\to 0^-} e^{\frac{1}{z}} = 0.$$

Since these limits differ, there is a no continuous extension and thus cannot be a removable singularity. Also $\lim_{z\to 0^-} f(z) = 0$ rules out $\lim_{z\to 0} |f(z)| = \infty$ and so it is not a pole.

Proposition 10.1.6. *If z_0 is an isolated singularity of $f(z)$,*

(a) z_0 is removable $\iff \lim\limits_{z\to z_0} (z - z_0)f(z) = 0$.

(b) z_0 is a pole $\iff$ it is not removable and $\lim\limits_{z\to z_0} (z - z_0)^{n+1} f(z) = 0$ for some $n \in \mathbb{Z}_{>0}$.

Proof.

(a) If z_0 is removable, let $g(z)$ be a holomorphic extension of $f(z)$ to z_0.

$$\begin{aligned}\lim_{z\to z_0}(z-z_0)f(z) &= \lim_{z\to z_0}(z-z_0)g(z)\\ &= 0g(z_0)\\ &= 0.\end{aligned}$$

Conversely let

$$h(z)=\begin{cases}(z-z_0)^2f(z) & z\neq z_0\\ 0 & z=z_0.\end{cases}$$

Let's show that it is differentiable at z_0:

$$\begin{aligned}h'(z_0) &= \lim_{z\to z_0}\frac{h(z)-h(z_0)}{z-z_0}\\ &= \lim_{z\to z_0}\frac{(z-z_0)^2f(z)-0}{z-z_0}\\ &= \lim_{z\to z_0}(z-z_0)f(z)\\ &= 0 \qquad\qquad \text{(by assumption)}.\end{aligned}$$

So actually $h(z)$ is holomorphic at z_0 and so we can ask to what order it vanishes.

$$h(z)=(z-z_0)^m k(z)$$

with $k(z)$ holomorphic at z_0 and $k(z_0)\neq 0$.

Claim. $m\geq 2$.

Proof. Since $h(z_0)=0$, we must have $m\geq 1$ ($m=0$ gives $0=h(z_0)=k(z_0)\neq 0$). Since $h'(z_0)=0$, we get

$$0=m(z_0-z_0)^{m-1}k(z_0)+(z_0-z_0)^m k'(z_0),$$

this is only zero if $m-1\geq 1$, i.e. $m\geq 2$. □

Then $f(z)=(z-z_0)^{m-2}k(z)$ is a holomorphic extension of $f(z)$.

(b) If $\lim_{z\to z_0}|f(z)|=\infty$, then for some $\epsilon>0$ and $0<|z-z_0|<\epsilon$, $|f(z)|>1$. By working close enough to z_0, we can assume $f(z)\neq 0$, hence $\frac{1}{f(z)}$ is holomorphic for $0<|z-z_0|<\epsilon$. But then

$$\lim_{z\to z_0}|f(z)|=\infty \implies \lim_{z\to z_0}\frac{1}{f(z)}=0.$$

So
$$\lim_{z\to z_0}(z-z_0)\frac{1}{f(z)} = 0\cdot 0 = 0.$$
Hence z_0 is a removable singularity of $\frac{1}{f(z)}$.
$$\frac{1}{f(z)} = (z-z_0)^n \ell(z)$$
where $\ell(z_0)\neq 0$ and $\ell(z)$ is holomorphic at $z=z_0$. Then
$$f(z) = (z-z_0)^{-n}\frac{1}{\ell(z)}.$$
So
$$\begin{aligned}\lim_{z\to z_0}(z-z_0)^{n+1}f(z) &= \lim_{z\to z_0}(z-z_0)\frac{1}{\ell(z)}\\ &= 0\cdot\frac{1}{\ell(z_0)}\\ &= 0.\end{aligned}$$
Suppose $\lim\limits_{z\to z_0}(z-z_0)^{n+1}f(z)=0$ and assume n is as small as possible. Let $p(z)=(z-z_0)^n f(z)$, then
$$\lim_{z\to z_0}(z-z_0)p(z) = 0.$$
So z_0 is a removable singularity of $p(z)$. Hence, $p(z)$ extends to a function holomorphic at z_0. Therefore,
$$\lim_{z\to z_0}(z-z_0)^n f(z)$$
exists $(=p(z_0))$ and it is nonzero by minimality of n. Then
$$\begin{aligned}\lim_{z\to z_0}|f(z)| &= \lim_{z\to z_0}|p(z)(z-z_0)^{-n}|\\ &= \lim_{z\to z_0}|p(z)|\cdot\lim_{z\to z_0}|(z-z_0)^{-n}|\\ &= |p(z_0)|\cdot\infty\\ &= \infty.\end{aligned}$$
Thus, $f(z)$ has a pole at z_0.

Definition 10.1.7 (Order)**.** This number n is the order of the pole of $f(z)$ at z_0.

□

Proposition 10.1.8. *Suppose z_0 is an isolated singularity of $f(z)$ and*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

near z_0, then

(a) *z_0 is removable $\iff a_k = 0$ for $k < 0$*

(b) *z_0 is a pole of order $n \iff a_{-n} \neq 0$ and $a_k = 0$ for $k < -n$.*

(c) *z_0 is essential $\iff a_{-n} \neq 0$ for infinitely many $n > 0$.*

Proof. (a) If there are no negative index terms, the Laurent series is a power series which defines a holomorphic function at z_0. If $f(z)$ is holomorphic at z_0, it has a power series about z_0, which is the same thing as a Laurent series with no negative index terms.

(b) If $f(z)$ has a pole of order n, $f(z) = (z - z_0)^{-n} g(z)$ for some $g(z)$ that is holomorphic at z_0 ($g(z_0) \neq 0$) (since $\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$, $(z - z_0)^n f(z)$ has a removable singularity at z_0, call resulting function $g(z)$). Write

$$g(z) = \sum_{k \geq 0} a_k (z - z_0)^k$$

with $a_0 = g(z_0) \neq 0$ (power series about z_0) so

$$f(z) = (z - z_0)^{-n} g(z) = \sum_{k \geq 0} a_k (z - z_0)^{k-n}.$$

The coefficient of $(z - z_0)^{-n}$ is $a_0 \neq 0$ and all lower coefficients are zero. Conversely, if

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$$

with $a_{-n} \neq 0$, then

$$(z - z_0)^{n+1} f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^{n+k+1}.$$

This is a power series with constant term zero. Thus, this defines a holomorphic function where value at z_0 is zero.

(c) Automatic if neither (a) nor (b) apply. We have an essential singularity, but neither of (a) and (b) applying means that infinitely many negative index terms are nonzeros. □

Chapter 11

Residue Theorem

11.1 Residue

Recall if $f(z)$ has an isolated singularity at z_0,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

with

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0)^{k+1}} dw.$$

In particular, when $k = -1$,

$$2\pi i a_{-1} = \oint_\gamma f(w) dw.$$

So if we can find a_{-1}, we can find the integral.

Definition 11.1.1 (Residue)**.** The coefficient a_{-1} in the Laurent series of $f(z)$ at z_0 is called the *residue* of $f(z)$ at z_0. We write this $\operatorname*{Res}_{z=z_0} f(z)$.

So we can state the best version of the Cauchy integral formula.

11.2 Residue Theorem

Theorem 11.2.1 (Residue Theorem)**.** *Let γ be a closed curve and $f(z)$ be a holomorphic function on γ and also on the region enclosed by γ except for isolated singularities. Then there are finitely many singularities inside the enclosed region and*

$$\oint_\gamma f(z) dz = 2\pi i \sum_k \underset{z=z_k}{Res} f(z).$$

We are summing over singularities enclosed by γ.

Proof. Let S be the set of singularities inside the enclosed region. Note that S is bounded (it is contained in the enclosed region, which is bounded by the Jordan curve theorem). Note that S is also closed (since S is the complement of the set of points where $f(z)$ is holomorphic. If $f(z)$ is holomorphic at z_0, it is differentiable on a disk centered at z_0, and so it is holomorphic on a disk centered at z_0. So the set of points where $f(z)$ is holomorphic is open, and its complement is closed).

Suppose there are infinitely many points in S. Consider a sequence of distinct points in S: $z_1, z_2, z_3, \ldots$. Then by Bolzano-Weierstrass theorem, which states that a bounded sequence has a convergent subsequnce, we know that there exists a subsequence

$$z_{i_1}, z_{i_2}, z_{i_3}, \ldots \to w.$$

Because S is closed, the limit point w must still be in S.

Claim. W is not an isolated singularity.

Proof. For any $\epsilon > 0$, there is some z_{i_j} within ϵ of w (because $z_{i_j} \to w$, it must get arbitrarily close to w), so f cannot be holomorphic on $0 < |z - w| < \epsilon$. Thus, there can be only finitely many points in S. □

Now draw little circles around singularities in γ and connect them to the boundary with line segments. Then

$$0 = \oint_{\text{new curve}} = \oint_\gamma + \sum \int_{\text{line segments}} + \sum \oint_{\text{little circles clockwise}}.$$

□

Example 11.2.2. Compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} dx.$$

Note if $f(z) = \frac{g(z)}{h(z)}$ with $g(z), h(z)$ holomorphic. $f(z)$ has singularities when $h(z) = 0$. But we showed that the zeros of a non-constant holomorphic function are isolated, which implies singularities of $f(z)$ are isolated. We take $g(z) = (z - z_0)^a \alpha(z)$ and $h(z) = (z - z_0)^b \beta(z)$ such that

$$f(z) = (z - z_0)^{a-b} \frac{\alpha(z)}{\beta(z)}.$$

$\alpha(z_0) \neq 0, \beta(z_0) \neq 0 \implies \frac{\alpha(z_0)}{\beta(z_0)} \neq 0$. $f(z)$ has a zero of order $a - b$ if $a > b$ and has a pole of order $b - a$ if $a < b$. So the singularities are poles (essential singularities).

Remark. If all singularities of $f(z)$ are isolated poles (not essential singularities). Then $f(z)$ is called a *meromorphic* function. All meromorphic functions are of the form $\frac{g(z)}{h(z)}$ with g, h holmorphic.

Answer. Let γ be the upper-plane semicircle anticlockwise with radius R. The singular points are $z = \pm i$, but $z = i$ is inside γ. Then by Residue theorem,

$$\int_{-R}^{R} \frac{1}{(1+x^2)^n} dx + \int_{\text{arc}} \frac{1}{(1+x^2)^n} dx = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^n}.$$

By ML-lemma,

$$\begin{aligned}
\left| \int_{\text{arc}} \frac{1}{(z^2+1)^n} dz \right| &\leq \pi R \text{max} \left(\frac{1}{|z^2+1|^n} \right) \\
&= \frac{\pi R}{\min |z^2+1|^n} \\
&\leq \frac{\pi R}{\min (|z|^2-1)^n} && \text{(reverse triangle inequality)} \\
&= \frac{\pi R}{(R^2-1)^n} \to 0.
\end{aligned}$$

So we only need to compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} dx = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^n}.$$

Consider when $n = 1$:

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}.$$

$$\begin{aligned}
\frac{1}{z+i} &= \frac{1}{(z-i)+2i} \\
&= \frac{1}{2i} \frac{1}{1+\frac{z-i}{2i}} \\
&= \frac{1}{2i} \sum_{k \geq 0} \left(-\frac{z-i}{2i} \right)^k.
\end{aligned}$$

So

$$\frac{1}{z^2+1} = \frac{1}{2i} \sum_{k \geq 0} \left(-\frac{1}{2i} \right)^k (z-i)^{k-1}.$$

The coefficient of $(z-i)^{-1}$ (i.e. the residue) comes from the $k = 0$ term,

$$\frac{1}{2i} \left(-\frac{1}{2i} \right)^0 = \frac{1}{2i}.$$

So the integral is

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = 2\pi i \frac{1}{2i} = \pi.$$

Now for $n = 2$, we have

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2} \frac{1}{(z+i)^2}.$$

$$\begin{aligned} \frac{1}{(z+i)^2} &= \left[\frac{1}{2i} \sum_{k \geq 0} \left(-\frac{z-i}{2i} \right)^k \right] \\ &= \left(\frac{1}{2i} \right)^2 + 2 \frac{1}{2i} \frac{1}{2i} \left(-\frac{z-i}{2i} \right) + \text{ higher order terms.} \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{(z^2+i)^2} &= \frac{1}{(z-i)^2} \left[\frac{1}{(2i)^2} + 2 \frac{1}{(2i)^2} \left(-\frac{z-i}{2i} + \cdots \right) \right] \\ &= \frac{1}{(2i)^2} \frac{1}{(z-i)^2} + \frac{2}{(2i)^2} \left(\frac{1}{-2i} \right) (z-i)^{-1} + \cdots . \end{aligned}$$

The residue is $\frac{2}{(2i)^2} \frac{1}{-2i} = \frac{2}{8i} = \frac{1}{4i}$. Thus,

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx = 2\pi i \frac{1}{4i} = \frac{\pi}{2}.$$

Recall the Binomial Theorem:

$$(1+z)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

If n is replaced by α not necessarily positive integer, then

$$(1+z)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} z^k = \sum_{k \geq 0} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k(k-1)\cdots 1} z^k.$$

Write

$$(1+z)^\alpha = e^{\alpha \log(1+z)},$$

which is holomorphic for $|z| < 1$. To compute Taylor series, differentiate $(1+z)^\alpha$ n times:

$$\begin{aligned} &\alpha(1+z)^{\alpha-1} \\ &\alpha(\alpha-1)(1+z)^{\alpha-2} \\ &\vdots \\ &\alpha(\alpha-1)\cdots(\alpha-k+1)(1+z)^{\alpha-k}\Big|_{z=0} \end{aligned}$$

$$\sum_{k\geq 0} \frac{f^{(k)}(z)}{k!} z^k = \sum_{k\geq 0} \binom{\alpha}{k} z^k$$

which converges for $|z| < 1$.

$$\begin{aligned}
\frac{1}{(z+i)^n} &= \frac{1}{(z-i+2i)^n} \\
&= \frac{1}{(2i)^n} \frac{1}{\left(1+\frac{z-i}{2i}\right)^n} \\
&= \frac{1}{(2i)^n} \left(1+\frac{z-i}{2i}\right)^{-n}.
\end{aligned}$$

Apply binomial theorem with $\alpha = -n$ to get

$$\begin{aligned}
&\frac{1}{(2i)^n} \sum_{k\geq 0} \binom{-n}{k} \left(\frac{z-i}{2i}\right)^k \\
&\qquad = \sum_{k\geq 0} \frac{1}{(2i)^n} \frac{1}{(2i)^k} \binom{-n}{k} (z-i)^k.
\end{aligned}$$

Thus,

$$\frac{1}{(z^2+1)^n} = \sum_{k\geq 0} \frac{1}{(2i)^{n+k}} \binom{-n}{k} (z-i)^{k-n}.$$

The residue is the coefficient of $(z-i)^{-1}$ which corresponds to $k = n-1$ and so it is

$$\frac{1}{(2i)^{n+(n-1)}} \binom{-n}{n-1} = \frac{1}{(2i)^{2n-1}} \binom{-n}{n-1}.$$

$$\begin{aligned}
\binom{-n}{n-1} &= \frac{(-n)(-n-1)\cdots - 2n+2}{(n-1)!} \\
&= (-1)^{n-1} \frac{n(n+1)\cdots(2n-2)}{(n-1)!} \\
&= (-1)^{n-1} \frac{(2n-2)(2n-3)\cdots}{(n-1)!} \frac{(n-1)(n-2)\cdots}{(n-1)!} \\
&= (-1)^{n-1} \binom{2n-2}{n-1}.
\end{aligned}$$

So

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^n}dx &= 2\pi i\frac{1}{(2i)^{2n-1}}(-1)^{n-1}\binom{2n-2}{n-1}\\ &= \pi\frac{1}{(2i)^{2n-2}}i^{2n-2}\binom{2n-2}{n-1}\\ &= \frac{\pi}{2^{2n-2}}\binom{2n-2}{n-1}\\ &= \frac{\pi}{4^{n-1}}\binom{2n-2}{n-1}.\end{aligned}$$

Definition 11.2.3 (Laplace transform)**.** Given $f : (0,\infty) \to \mathbb{C}$, the *Laplace transform* of f is

$$\mathcal{L}(f)(s) = \int_0^{\infty} f(t)e^{-st}dt.$$

Here $t \in \mathbb{R}^+$ and $s \in \mathbb{C}$. Note that this might not converge for all s.

Example 11.2.4. $f(t) = \cos(\omega t)$ where $\omega \in \mathbb{R}^+$. Then

$$f(t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right).$$

Then

$$\begin{aligned}\mathcal{L}(f)(s) &= \int_0^{\infty}\frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)e^{-st}dt\\ &= \frac{1}{2}\int_0^{\infty} e^{-(s-i\omega)t}dt + \frac{1}{2}\int_0^{\infty} e^{-(s+i\omega)t}dt\\ &= \frac{1}{2}\frac{1}{-(s-i\omega)}e^{-(s-i\omega)t}\Big|_{t=0}^{\infty} + \frac{1}{2}\frac{1}{-(s+i\omega)}e^{-(s+i\omega)t}\Big|_{t=0}^{\infty}.\end{aligned}$$

Note that

$$\left|e^{-(s+i\omega)t}\right| = e^{\Re(s)t}.$$

Then if $\Re(s) > 0$,

$$\lim_{t\to\infty} e^{-(s\pm i\omega)t} = 0.$$

So if $\Re(s) > 0$, we get

$$\begin{aligned}\frac{1}{2}\left[0-\frac{1}{-(s-i\omega)}\right]+\frac{1}{2}\left[0-\frac{1}{-(s+i\omega)}\right] &= \frac{1}{2}\left(\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right)\\ &= \frac{s}{s^2+\omega^2}.\end{aligned}$$

Conclusion: if $\Re(s) > 0$ and $s \neq \pm i\omega$,

$$\mathcal{L}(f)(s) = \frac{s}{s^2 + w^2}.$$

Next we will compute the inverse Laplace transform.

Definition 11.2.5 (Inverse Laplace transform)**.** The *inverse Laplace transform* is defined by

$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} ds$$

where $t \in \mathbb{R}^+$ and $s \in \mathbb{C}$.

Remark. γ is a line parallel to the imaginary axis and c is any real number large enough such that all singularities of $F(s)$ are to the left of this line.

We will check that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s}{s^2 + \omega^2} e^{st} ds = \cos(\omega t).$$

To do so, we will use the residue theorem. We need a closed curve to use the theorem and we will integrate around a rectangle with vertices $c \pm iT$, and $-d \pm iT$. As $T \to \infty$, the right side becomes the integral we want. The singularities are at $\pm i\omega$, so we can take c to be any positive real numbers.

We will eventually make $T, d \to \infty$. Let's bound the integrals we have using ML-lemma:

$$\left| \int_{c+iT}^{-d+iT} \frac{s}{s^2 + \omega^2} e^{st} ds \right| \leq (c + d) \max_{-d \leq r \leq c} \left| \frac{r + iT}{(r + iT)^2 + \omega^2} e^{(r+iT)t} \right|.$$

Note that $|r + iT| \leq |-d + iT| = \sqrt{d^2 + T^2}$. Then

$$\frac{1}{|(r + iT)^2 + \omega^2|} = \frac{1}{|r^2 - T^2 + \omega^2 + 2riT|} \leq \frac{1}{|r^2 - T^2 + \omega^2|}.$$

We'll take $d = \sqrt{T}$, then $|r^2| \leq T$, so as $T \to \infty$, $|r^2 - T^2 + \omega^2| = T^2 - r^2 - \omega^2$. So our bound will be

$$\frac{1}{T^2 - d^2 - \omega^2} = \frac{1}{T^2 - T - \omega^2}.$$

$$\begin{aligned} \left| e^{(r+iT)t} \right| &= e^{\Re((r+iT)t} \\ &= e^{rt} \leq e^{ct}. \end{aligned}$$

$$
\begin{aligned}
\left| \int_{c+iT}^{-\sqrt{T}+iT} \right| &\le \frac{(c+\sqrt{T})\sqrt{T^2+T}}{T^2-T-\omega^2} \\
&= \frac{T\sqrt{T}\left(\frac{c}{\sqrt{T}}+1\right)\sqrt{1+\frac{1}{T}}}{T^2\left(1-\frac{1}{T}-\frac{\omega^2}{T^2}\right)},
\end{aligned}
$$

which goes to 0 as $T \to \infty$. Exactly the same bound applies for the integral $\int_{-\sqrt{T}-iT}^{c-iT}$.

$$
\left| \int_{-d+iT}^{-d-iT} \frac{s}{s^2+\omega^2} e^{st} ds \right| \le 2T \max_{-T \le r \le T} \left| \frac{(-d+ir)}{(-d+ir)^2+\omega^2} e^{(-d+ir)t} \right|.
$$

Note that $|-d+ir| = \sqrt{d^2+r^2} \le \sqrt{d^2+T^2}$ and

$$
\begin{aligned}
\frac{1}{|(-d+ir)^2+\omega^2|} &\le \frac{1}{|(-d+ir)^2|-\omega^2} \\
&\le \frac{1}{d^2-\omega^2} \\
&= \frac{1}{T-\omega^2}.
\end{aligned}
$$

$$
|e^{(-d+ir)t}| = e^{\Re((-d+ir)t)} = e^{-dt} = e^{-\sqrt{T}t}.
$$

Thus,

$$
\begin{aligned}
\left| \int_{-\sqrt{T}+iT}^{-\sqrt{T}-iT} \right| &\le \frac{2T\sqrt{T^2+T}}{T-\omega^2} e^{-\sqrt{T}t} \\
&= \frac{T^2}{T} e^{-t\sqrt{T}} \frac{2\sqrt{1+\frac{1}{T}}}{1-\frac{\omega^2}{T}},
\end{aligned}
$$

which goes to zero as $T \to \infty$ since

$$
\lim_{T\to\infty} T e^{-t\sqrt{T}} = 0.
$$

So

$$
\int_{c-i\infty}^{c+i\infty} = 2\pi i \left(\operatorname*{Res}_{s=i\omega} \frac{s}{s^2+\omega^2} e^{st} + \operatorname*{Res}_{s=-i\omega} \frac{s}{s^2+\omega^2} e^{st} \right).
$$

We can rewrite as

$$
\frac{s}{(s-i\omega)(s+i\omega)} e^{st},
$$

where residue at $i\omega$ is $\frac{i\omega}{i\omega+i\omega}e^{i\omega t}$ and residue at $-i\omega$ is $\frac{-i\omega}{-i\omega-i\omega}e^{-i\omega t}$. Thus,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} = \frac{1}{2}e^{i\omega t} + \frac{1}{2}e^{-i\omega t} = \cos(\omega t) = f(t).$$

We can also use the residue theorem to solve some problems in discrete math.

Example 11.2.6. How many ways are there to express k as a sum using some given numbers $a, b, \ldots$? In other words, how many solutions are there to

$$ax_1 + bx_2 + \cdots = k,$$

where $x_1, x_2, \ldots$ are non-negative integers. It turns out that the largest number that cannot be written using a non=negative integer linear combination of a, b is $ab - a - b$. Now we are interested in knowing how to find an expression for number of ways to express k using a and b.

The number of ways to write k as a sum using only a is 1 if k is a multiple of a and 0 otherwise.

Let's write a power series where the coefficient of z^k is the number of ways to write k as a sum of a's:

$$\begin{aligned} 1 \cdot z^0 + 0 \cdot z^1 + 0 \cdot z^2 + \cdots 1 \cdot z^a + 0 \cdot z^{a+1} + \cdots &= z^0 + z^a + z^{2a} + z^{3a} + \cdots \\ &= \frac{1}{1 - z^a}. \end{aligned}$$

Similarly $\frac{1}{1-z^b}$ has a power series whose coefficients count how many ways k can be written as a sum of b's. Then

$$\frac{1}{1-z^a}\frac{1}{1-z^b} = \sum_{k \geq 0} a_k z^k$$

where a_k is the number of ways of expressing k as a sum of a's and b's. Why? To see this, we expand the power series

$$\begin{aligned} (1 + z^a + z^{2a} + \cdots + z^{xa} + \cdots)(1 + z^b + z^{2b} + \cdots + z^{yb} + \cdots) &= \sum_{x,y \geq 0} z^{xa} z^{yb} \\ &= \sum_{x,y \geq 0} z^{ax+by}. \end{aligned}$$

The number of ways z^k appears is equal to the number of pairs (x, y) such that $x, y \geq 0$ and $ax + by = k$. So to know whether k can be written in this form, it is equivalent to know a_k in the power series:

$$\frac{1}{1-z^a}\frac{1}{1-z^b} = \sum_{k\geq 0} a_k z^k.$$

By the residue theorem, this equals

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{(1-z^a)(1-z)^b z^{k+1}} = a_k.$$

Here the integrand is $\sum_{n\geq 0} a_n z^{n-k-1}$ where the coefficient of z^{-1} corresponds to $n = k$, which is a_k. γ is a small closed curve around 0. We will actually compute the integral around a big circle Γ of radius R (which we will take to ∞). So

$$\frac{1}{2\pi i}\oint_\Gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}dz = \underbrace{\operatorname*{Res}_{z=0} f(z)}_{a_k} + \sum_{\omega^a=1,\omega^b=1} \operatorname*{Res}_{z=\omega} f(z).$$

By ML-lemma,

$$\begin{aligned}\left|\oint_\Gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}dz\right| &\leq 2\pi R\cdot \max_{|z|=R}\left|\frac{1}{(1-z^a)(1-z^b)z^{k+1}}\right| \\ &\leq \frac{2\pi R}{(R^a-1)(R^b-1)R^{k+1}} \to 0.\end{aligned}$$

Thus, since the integral converges to 0, we have

$$a_k = \operatorname*{Res}_{z=0} f(z) = -\sum_{\omega^a=1,\omega^b=1} \operatorname*{Res}_{z=\omega} f(z).$$

Now assume a, b are coprime, i.e. $\gcd(a, b) = 1$ because then the solution to $\omega^a = 1$ and $\omega^b = 1$ is $\omega = 1$. (follows from Bézout's identity: there exists integers x, y such that $ax + by = gcd(a, b)$ and so $1 = 1^x 1^y = (\omega^a)^x(\omega^b)^y = \omega^{ax+by} = \omega^{gcd(a,b)} = \omega^1 = \omega$.)
Conclusion: $f(z)$ has poles of order 1 at $e^{2\pi ir/a}$ for $r = 1, 2, \ldots, a-1$ and at $e^{2\pi is/b}$ for $s = 1, 2, \ldots, b-1$. We can compute these relatively easily. The pole at $z = 1$ has order 2 and so the residue calculation is trickier, which turns out to be $-\frac{a+b+2k}{2ab}$. Final expression turns out to be

$$a_k = \frac{a+b+2k}{2ab} + \frac{1}{a}\sum_{r=1}^{a-1}\frac{1}{e^{2\pi i\frac{r}{a}\cdot k}(1-e^{2\pi i\frac{r}{a}b})} + \frac{1}{b}\sum_{s=1}^{b-1}\frac{1}{e^{2\pi i\frac{s}{b}\cdot k}(1-e^{2\pi i\frac{s}{b}a})}.$$

The two sums are periodic in k with periods a, b respectively. When k is large, $\frac{k}{ab}$ is larger than all terms, and so $a_k \geq 0$ for k large enough.

11.3 Gamma Function

Instead of talking about things that apply to all holomorphic functions, we will take a look at a particular function that comes up somewhat often (and cannot be expressed using familiar functions like polynomials and exponentials). These functions are called *special functions.*

Definition 11.3.1 (Gamma function)**.** The *gamma function* is defined as

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx,$$

where $x^{s-1} = e^{(s-1)\ln(x)}$ is a holomorphic function of s. This converges for $\Re(s) > 0$:

$$\Gamma(s) = \int_0^1 x^{s-1} e^{-x} dx + \int_1^\infty x^{s-1} e^{-x} dx$$

where first integral converges only when $\Re(s) > 0$ whereas the second integral converges for any value of s, since exponentials grow much faster than polynomial.

For comparison:

$$\int_\epsilon^1 x^{s-1} dx = \frac{1}{s}(1 - \epsilon^s).$$

The limit as $\epsilon \to 0$ only exists when $\Re(s) > 0$.

Fact. For $\Re(s) > 0$, $\Gamma(s)$ is holomorphic. (from measure theory)

11.3.1 Properties of the gamma function

1. $\Gamma(s)$ is continuous.

2. $$\begin{aligned} \oint_\gamma \Gamma(s) ds &= \int_\gamma \int_0^\infty x^{s-1} e^{-x} dx ds \\ &= \int_0^\infty \oint_\gamma x^{s-1} e^{-x} ds dx \end{aligned}$$

   i.e., we can swap order of integration and the inner integration is zero by Cauchy's theorem, so

   $$\oint_\gamma \Gamma(s) ds = 0$$

   and Morera's theorem applies, so $\Gamma(s)$ is holomorphic.

3. Integrate by parts, we have

$$\begin{aligned}\int_0^\infty x^{s-1}e^{-x}dx &= \frac{x^s}{s}e^{-x}\bigg|_{x=0}^{\infty} - \int_0^\infty \frac{x^s}{s}(-e^{-x})dx \\ &= 0 + \int_0^\infty \frac{x^s}{s}e^{-x}dx \\ &= \frac{1}{s}\Gamma(s+1).\end{aligned}$$

   Thus, $\Gamma(s+1) = s\Gamma(s)$.

4. We can compute that $\Gamma(1) = 1$, and by 3, we have $\Gamma(2) = 1 \cdot \Gamma(1) = 1$ and $\Gamma(3) = 2\Gamma(2) = 2$ and inductively we would have

$$\Gamma(n) = (n-1)!$$

   if $n \in \mathbb{Z}_{>0}$.

   Remark. We can view $\Gamma(s)$ as an extension of the factorial to complex numbers.

Remark. Unfortunately, for most s, the values of $\Gamma(s)$ can't be expressed in terms of simpler functions. For example $\Gamma(1/3) = 2.6789....$ However, we can find $\Gamma(1/2)$ using substitution $x = y^2$ (so $dx = 2ydy$). Then

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty x^{-\frac{1}{2}}e^{-x}dx \\ &= \int_0^\infty \frac{1}{y}e^{-y^2}2ydy \\ &= 2\int_0^\infty e^{-y^2}dy \\ &= \sqrt{\pi}.\end{aligned}$$

Then using this, we can compute for $\Gamma(\frac{2n+1}{2}$:

$$\begin{aligned}\Gamma\left(\frac{2n+1}{2}\right) &= \frac{2n-1}{2}\Gamma\left(\frac{2n-1}{2}\right) \\ &= \qquad \vdots \\ &= \frac{(2n-1)(2n-3)\cdots 1\cdot\sqrt{\pi}}{2^n} \\ &= \frac{(2n)!}{n!4^n}\sqrt{\pi}.\end{aligned}$$

Remark. $\Gamma(s) = \frac{1}{s}\Gamma(s+1)$ makes sense when $\Re(s) > -1$ and $s \neq 0$. This serves as a definition of $\Gamma(s)$. On this larger set:

$$\begin{aligned}
\Gamma(s) &= \frac{1}{s}\Gamma(s+1) \\
&= \frac{1}{s}\cdot\frac{1}{s+1}\Gamma(s+2) \qquad \text{(defined when } \Re(s) > -2 \text{ and } s \neq 0, -1\text{)} \\
&= \qquad \vdots \\
&= \frac{1}{s}\cdot\frac{1}{s+1}\cdot\dots\cdot\frac{1}{s+n-1}\Gamma(s+n).
\end{aligned}$$

So $\Gamma(s)$ extends to a function for $\Re(s) > -n$ and $s \neq 0, -1, -2, \dots, -(n-1)$. Thus, for any $s \in \mathbb{C}$ that is a non-positive integer, we can define

$$\Gamma(s) = \frac{1}{s\cdot(s+1)\cdots(s+n-1)}\Gamma(s+n)$$

for any n with $\Re(s) > -n$.

It also turns out that $\Gamma(s)$ is holomorphic on $\mathbb{C}\backslash\{0, -1, -2, \dots\}$. In fact, for $k \in \mathbb{Z}_{\geq 0}$, $\Gamma(s)$ has a pole of order 1 at $-k$ with residue $(-1)^k/k!$.

$$\begin{aligned}
\lim_{s\to -k}(s-(-k))\Gamma(s) &= \lim_{s\to -k}(s+k)\frac{1}{s(s+1)\cdots(s+k)}\Gamma(s+k+1) \\
&= \frac{1}{(-k)(-k+1)\cdots(-1)}\Gamma(-k+k+1) \\
&= \frac{(-1)^k}{k!}.
\end{aligned}$$

11.4 The Riemann Zeta function

Definition 11.4.1 (Zeta function)**.** The *zeta function* is

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots.$$

It converges for $\Re(s) > 1$ (because $|n^{-s}| = |e^{\ln(n^{-s})}| = |e^{-s\ln(n)}| = e^{-\Re(s)\ln(n)} = n^{-\Re(s)}$ and so using the integral test, $\zeta(s)$ converges when $s > 1$ and when $s = 1$, $\zeta(1)$ is a harmonic series, which diverges.)

One can check that for $\epsilon > 0$, $\sum_{n\geq 1} n^{-s}$ converges uniformly on $\Re(s) \geq 1+\epsilon$, this can be used to show that Morera's theorem applies and hence $\zeta(s)$ is holomorphic for $\Re(s) > 1$.

Remark. For $\Re(s) > 0$, we can rewrite

$$n^{-s} = s\int_0^\infty x^{-(s+1)}dx.$$

Using the above expression, we can write

$$\begin{aligned}
\zeta(s) &= \sum_{n\geq 1} n^{-s} \\
&= \sum_{n\geq 1} s\int_n^\infty x^{-(s+1)}dx \\
&= s\int_1^\infty \lfloor x\rfloor x^{-(s+1)}dx \\
&= s\int_1^\infty \frac{x-\{x\}}{x^{s+1}}dx \\
&= s\int_1^\infty \frac{x}{x^{s+1}}dx - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx \\
&= \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx.
\end{aligned}$$

The first term makes sense for $s \neq 1$ and the second part converges when $\Re(s) > 0$. So we can define $\zeta(s)$ for $\Re(s) > 0$ provided $s \neq 1$.

$$\begin{aligned}
\left|\int_1^\infty \frac{\{x\}}{x^{s+1}}dx\right| &\leq \int_1^\infty \frac{|\{x\}|}{|x^{s+1}|}dx \\
&\leq \int_1^\infty \frac{1}{x^{\Re(s)+1}}dx.
\end{aligned}$$

$\zeta(s)$ has a pole of order 1 with residue 1 at $s = 1$. Actually we also have

$$\zeta(s) = \frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-2)\zeta(1-s).$$

$\zeta(s)$ is famous for the connection to prime numbers.

Remark. Recall prime factorization:

$$n = \prod_{i=1}^k p_i^{a_i} \implies n^{-s} = \prod_{i=1}^k p_i^{-a_i}.$$

Then we have

$$\zeta(s) = \prod_p (1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots) = \prod_p \frac{1}{1-p^{-s}}.$$

By uniqueness of prime factorization, each n^{-s} arises exactly once. Then

$$\begin{aligned}
\ln(\zeta(s)) &= \sum_p -\ln(1-p^{-s}) \\
\frac{\zeta'(s)}{\zeta(s)} &= \sum_p -\frac{\ln(p)p^{-s}}{1-p^{-s}} \\
&= -\sum_p \left(\frac{\ln(p)}{p^s} + \frac{\ln(p)}{p^2 s} + \cdots\right) \\
&= \sum_{n\geq 1} -\frac{\Lambda(n)}{n^s}
\end{aligned}$$

where

$$\Lambda(n) = \begin{cases} \ln(p) & \text{if } n \text{ is a power of the prime } p \\ 0 & \text{otherwise.} \end{cases}$$

Fact. For $x > 0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} n^{-s}\frac{x^s}{s}ds = \begin{cases} 1 & x > n \\ 0 & x < n. \end{cases}$$

Thus,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}dx &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}\frac{x^s}{s}ds \\
&= \sum_{n\geq 1}\frac{\Lambda(n)}{2\pi i}\int_{c-i\infty}^{c+i\infty} n^{-s}\frac{x^s}{s}ds \\
&= \sum_{n<x}\Lambda(n).
\end{aligned}$$

Let $\pi(x)$ be the number of primes less than or equal to x. Then

$$\begin{aligned}\sum_{n\leq x}\Lambda(n) &= \sum_{p^k\leq x}\ln(p)\\ &= \sum_{p\leq x}\ln(p)\lfloor\log_p(x)\rfloor\\ &= \sum_{p\leq x}\ln(p)\lfloor\frac{\ln(x)}{\ln(p)}\rfloor\\ &\leq \sum_{p\leq x}\frac{\ln(x)}{\ln(p)}\\ &= \ln(x)\sum_{p\leq x}1\\ &= \ln(x)\pi(x).\end{aligned}$$

Thus,

$$\frac{\sum_{n\leq x}\Lambda(n)}{\ln(x)} \leq \pi(x).$$

In fact one can show

$$\lim_{x\to\infty}\frac{\frac{\sum_{n\leq x}\Lambda(n)}{\ln(x)}}{\pi(x)} = 1.$$

Hence, by residue theorem

$$\begin{aligned}\sum_{n\leq x}\Lambda(n) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}dx\\ &= x - \sum_{\zeta:\text{zeros of } \zeta(s) \text{ with } 0\leq\Re(\zeta)\leq 1}\frac{x^\zeta}{\zeta} - \ln(2\pi) - \frac{1}{2}\ln(1-x^{-s})\end{aligned}$$

It turns out $\Re(\zeta) < 1$ for all ζ, so x is the dominant term and thus

$$\sum_{n\leq x}\Lambda(n) \approx x.$$

Finally, we have

$$\pi(x) \approx \frac{x}{\ln(x)},$$

which is the Prime number theorem. Riemann Hypothesis: $\Re(\zeta) = \frac{1}{2}$.