CS70
Graph Theory

Kelvin Lee

UC Berkeley

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Overview

Basic Definitions

Definition (Graph)

A **graph** G is defined by a set of vertices V and a set of edges E, denoted by $G = (V, E)$. There are directed and undirected graphs.

Definition (Edge)

We use $(u, v) \in E$ to denote an edge from vertex u to vertex v. If the graph is undirected then we have $(u, v), (v, u) \in E$.

Definition (Degree)

The **degree** of a vertex v is the number of edges that are incident to v. A vertex with degree 0 is an *isolated* vertex.

Paths, Cycles, Walks, Tours

Definition (Path)

A **path** in $G = (V, E)$ is a sequence of edges $\{(v_i, v_{i+1})\}_{i=1}^{k-1}$ where each vertex are distinct (no repeated vertex).

Definition (Cycle)

A **cycle** is simply a path with $v_1 = v_k$ (no repeated vertex).

Definition (Walk)

A **walk** is a sequence of edges with possible repeated vertex or edge.

Definition (Tour)

A **tour** is a walk that with $v_1 = v_k$ (with possibly repeated vertex).

Handshaking Theorem

Theorem (Handshaking)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$\sum_{v \in V} \deg(v) = 2m.$$

Connectivity

Definition (Connected)

A graph is **connected** if there is a path between any two distinct vertices.

Definition (Connected Components)

A **connected component** is a maximal set of connected nodes in a graph.

Eulerian Tour

Definition (Eulerian Tour)

An **Eulerian Tour** is a tour that visits each edge exactly once.

Theorem (Euler's Theorem)

An undirected graph $G = (V, E)$ has an Eulerian tour iff all vertices have even degree and are connected.

Complete Graphs

Definition (Complete graph)

A graph G is **complete** if each pair of its vertices is connected by an edge. We use K_n to denote a complete graph on n vertices.

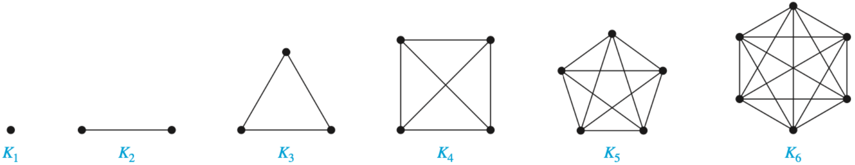


Figure: Examples of complete graphs.

A complete graph has $|V|(|V|-1)/2$ edges.

Bipartite Graphs

Definition (Bipartite)

A graph $G = (V, E)$ is **bipartite** if $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ such that vertices in V_1 are only connected by those in V_2. We use $K_{n,m}$ to denote a complete bipartite graph partitioned into n and m vertices.

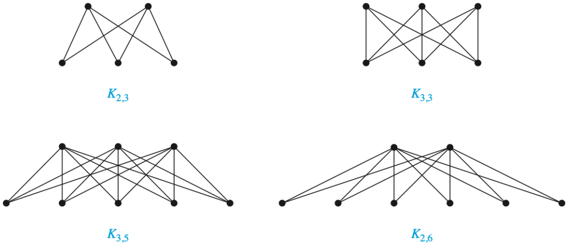


Figure: Complete bipartite graphs.

Planarity

Definition (Planar)

A graph is called **planar** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint).

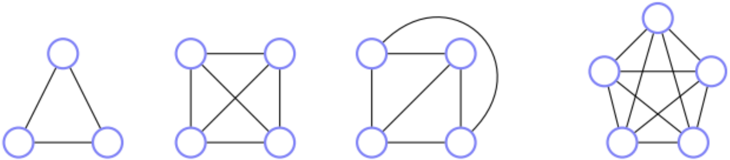


Figure: K_4 is planar because it can be drawn without crossing edges.

Euler's Formula

How to check for planarity?

Theorem (Euler's formula)

*For every connected **planar** graph with v vertices, f faces, and e edges,*

$$v + f = e + 2.$$

Corollary

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

This can be used to check that K_5 is non-planar.

Corollary

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no cycles of length three, then $e \geq 2v - 4$.

This can be used to check that $K_{3,3}$ is non-planar.

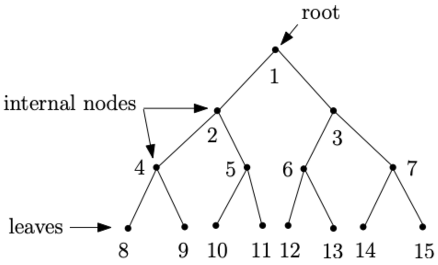
Non-planarity

Theorem (Kuratowski's Theorem)

A graph is ***non-planar*** *iff it contains K_5 or $K_{3,3}$*

To prove that a graph G is non-planar, show that you can find K_5 or $K_{3,3}$ as a subgraph in G.

Trees



Definition (Tree)

If G is a **tree**, then

- G is connected and contains no cycles.
- G is connected and has $|V| - 1$ edges.
- G is connected, and the removal of any single edge disconnects G.
- G has no cycles, and the addition of any single edge creates a cycle.

Trees

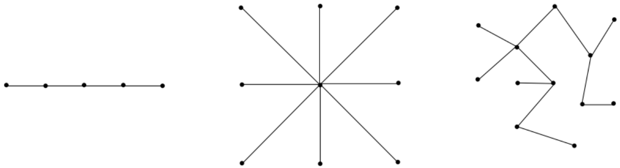


Figure: More examples of trees.

Theorem

G is connected and has $|V| - 1$ edges is equivalent to G is connected and contains no cycles.

Hypercubes

- An n-dimensional hypercube $G = (V, E)$ has $V = \{0, 1\}^n$, the set of all n-bit strings and each vertex is labeled by a unique n-bit string.
- Vertices $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$ are neighbors if and only if there is an $i \in \{1, \ldots, n\}$ such that $x_j = y_j$ for all $j \neq i$, and $x_i \neq y_i$.
- The n-dimensional hypercube has 2^n vertices.

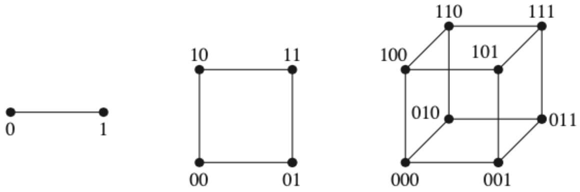


Figure: Hypercubes.

Hypercubes

Lemma

The total number of edges in an n-dimensional hypercube is $n2^{n-1}$.

Proof:

The degree of each vertex is n, since n bit positions can be flipped in any $x \in \{0, 1\}^n$. since each edge is counted twice, once from each endpoint, this yields a total of $n2^n/2 = n2^{n-1}$ edges. □

Problem Time!