CS70
Modular Arithmetic

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Overview

Basic Definitions

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In the modular space, the **multiplicative inverse** of x mod m is y if $xy \equiv 1 \pmod{m}$.

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Theorem (Existence of multiplicative inverse)

$gcd\,(x, m) = 1 \implies x$ *has a multiplicative inverse modulo m and it is* ***unique***.

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- When $\gcd(x, y) = 1$, we can deduce that b is an inverse of y mod x.
- This uses back substitutions repetitively so that the final expression is in terms of x and y.

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- A is the **domain** and B is the **co-domain**.
- Pre-image is a **subset** of domain, and the image/range is the **subset** of co-domain.
  - If $f(a) = b$, where $a \in A$ and $b \in B$, then we say that b is the **image** of a and a is the **pre-image** of b.

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A function f is called **onto**, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$. We also say that f is **surjective** if it's onto.

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- To show that a function is *onto*, choose $a = f^{-1}(b)$ and so $f(f^{-1}(b)) = b$.

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- If $f : A \to B$ is a bijection, it will have an **inverse** function (a lemma from notes), and $|A| = |B|$.

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where $N = \prod_{i=1}^{k} n_i$ and $b_i = \frac{N}{n_i}\left(\frac{N}{n_i}\right)_{n_i}^{-1}$ where $\left(\frac{N}{n_i}\right)_{n_i}^{-1}$ denotes the multiplicative inverse $(\bmod\, n_i)$ of the integer $\frac{N}{n_i}$.

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To see why x is a solution, notice that for each $i = 1, 2, \ldots, k$, we have

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3. For each $i = 1, 2, \ldots, k$, compute $z_i \equiv y_i^{-1} \bmod n_i$ (z_i exists since $n_1, n_2, \ldots, n_k$ are pairwise coprime).

Chinese Remainder Theorem

General construction:

1. Compute $N = n_1 \times n_2 \times \cdots \times n_k$.
2. For each $i = 1, 2, \ldots, k$, compute
$$y_i = \frac{N}{n_i} = n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_k.$$
3. For each $i = 1, 2, \ldots, k$, compute $z_i \equiv y_i^{-1} \bmod n_i$ (z_i exists since $n_1, n_2, \ldots, n_k$ are pairwise coprime).
4. Compute
$$x = \sum_{i=1}^{k} a_i y_i z_i$$
and $x \bmod N$ is the unique solution modulo N.

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2. Re-write this modulus as an equation, $x = j_k n_k + a_k$, for some positive integer j_k.
3. Substitute the expression for x into the congruence with the next largest modulus, $x \equiv a_k \pmod{n_k} \implies j_k n_k + a_k \equiv a_{k-1} \pmod{n_{k-1}}$.

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4. Solve this congruence for j_k.

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5. Write the solved congruence as an equation, and then substitute this expression for j_k into the equation for x.

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4. Solve this congruence for j_k.
5. Write the solved congruence as an equation, and then substitute this expression for j_k into the equation for x.
6. Continue substituting and solving congruences until the equation for x implies the solution to the system of congruences.

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- Start with mod 7. Write $x = 7k + 6$.

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- Start with mod 7. Write $x = 7k + 6$.
- Then we have $7k + 6 \equiv 4 \pmod{5} \implies k \equiv 4 \pmod{5}$.

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- Then we have $7k + 6 \equiv 4 \pmod 5 \implies k \equiv 4 \pmod 5$.
- Then solving for k gives $5j + 4$.

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- Now we have $x = 7k + 6 = 7(5j + 4) + 6 = 35j + 34$.

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- Then we have $7k + 6 \equiv 4 \pmod 5 \implies k \equiv 4 \pmod 5$.
- Then solving for k gives $5j + 4$.
- Now we have $x = 7k + 6 = 7(5j + 4) + 6 = 35j + 34$.
- Then $35j + 34 \equiv 1 \pmod 3 \implies j \equiv 0 \pmod 3 \implies j = 3t$.

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Solution:

- Start with mod 7. Write $x = 7k + 6$.
- Then we have $7k + 6 \equiv 4 \pmod{5} \implies k \equiv 4 \pmod{5}$.
- Then solving for k gives $5j + 4$.
- Now we have $x = 7k + 6 = 7(5j + 4) + 6 = 35j + 34$.
- Then $35j + 34 \equiv 1 \pmod{3} \implies j \equiv 0 \pmod{3} \implies j = 3t$.
- Finally, we have $x = 35(3t) + 34 = 105t + 34 \implies x \equiv \boxed{34} \pmod{105}$.

Problem Time!