# CS70 Geometric and Poisson Distributions

Kelvin Lee

UC Berkeley

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# Overview

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$$e^x = \sum_{i=1}^{\infty} \frac{x^i}{i!}.$$

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$$\mathrm{Var}(X) = \lambda.$$

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# Problem Time!