CS70
Continuous Probability

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Overview

Probability Density Function

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$$f(x) = \frac{dF(x)}{dx}.$$

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We interpret $f(x, y)$ as the **probability per unit area** in the vicinity of (x, y).

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Problem Time!