CS70 Continuous Probability II

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Overview

1. Normal Distribution
2. Normal Random Variables Standardization
3. Sum of Independent Normal Random Variables
4. Central Limit Theorem

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Remark: The CDF of a normal random variable is symmetrical, so

$$\Phi(-x) = 1 - \Phi(x).$$

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by a simple change of variable $x = \sigma y + \mu$ in the integral. Hence Y is standard normal, which is obtained from X by shifting the origin to μ and scaling by σ. □

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- CLT is stronger; it states that the distribution of the sample mean converges to normal distribution (this works for any sampling distribution)!

Problem Time!