CS70

Finite Markov Chains

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Overview

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$$\mathbb{P}(X_{n+1} = j \mid X_0, \ldots, X_n = i) = \mathbb{P}(i, j) \qquad i, j \in \mathcal{X}.$$

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A finite irreducible Markov chain has an unique invariant distribution.

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Theorem

Let $\{X_n\}$ be an irreducible and aperiodic Markov Chain with invariant distribution π. Then for all $i \in \mathcal{X}$,

$$\lim_{n \to \infty} \pi_n(i) = \pi(i).$$

Problem Time!