CS70
Countability

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Overview

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- To show that two infinite sets have the same **cardinality**, we need to establish a bijection (one-to-one correspondence) between the two sets.

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\mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & \dots & & \\
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- We have just shown that $\infty + 1 = \infty$!

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This function is in fact a bijection. Thus, the two sets have the same size.

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- The examples we did earlier are countable because they are subsets of $\mathbb{N}$, which is a **countable** set.

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Definition 2

If there is a surjective function $f : A \to B$, then $|A| \geq |B|$.

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Theorem (Schröder–Bernstein Theorem (Cantor–Bernstein))

If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection h between A and B.

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- This theorem will be very useful when showing a set S is countable. We can give separate injections $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$, instead of designing a bijection (which is trickier).

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- However, not all points are valid.
- Thus, we can actually tell that $|\mathbb{Z} \times \mathbb{Z}| \geq |\mathbb{Q}|$.
- If we are able to come up with an injection from $\mathbb{Z} \times \mathbb{Z}$ to N, then this will also be an injection from $\mathbb{Q}$ to $\mathbb{N}$ (why?).

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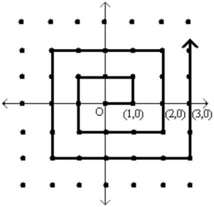

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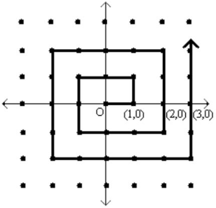

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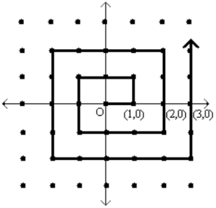

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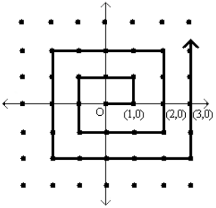

- Map each pair (a, b) to its position along the spiral, starting at the origin(indexing through each point starting from index 0).
- This mapping maps every pair of integers injectively to a natural number.
- Thus we have $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| \leq |\mathbb{N}|$. Remember that $|\mathbb{N}| \leq |\mathbb{Q}|$, then by the Cantor-Bernstein Theorem $|\mathbb{N}| = |\mathbb{Q}|$.

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- One example would be $\mathbb{R}[0, 1]$ demonstrated in lecture.
- We can create a real number where each of its ith digit differs from the ith digit of the ith element.
- Thus the real interval $\mathbb{R}[0, 1]$ is uncountable, so do its supersets.

Problem Time!