CS70
Discrete Probability II

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Overview

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This is exactly the **Law of Total Probability**, which is a very important law in probability theory.

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Pairwise independence does not imply mutual independence!

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$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2) \cdot \dots \cdot P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right).$$

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$$= \sum_{i=1}^{n} P\left(A_i\right) - \sum_{i<j} P\left(A_i \cap A_j\right) + \ldots + (-1)^{n-1} P\left(A_1 \cap A_2 \cap \cdots \cap A_n\right).$$

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  Summing up all $\mathbb{P}(A_i)$ only overestimate the probability of the union (equality holds when they are disjoint).

Problem Time!