Continuous Probability Practice

Kelvin Lee

December 4, 2020

1 Problems

Problem 1 (Sp19 Final). Consider a standard Gaussian random variable Z whose PDF is

$$\forall z \in \mathbb{R}, \quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Define another random variable X such that $X = |Z|$.

(a) Determine a reasonably simple expression for $f_X(x)$, the PDF of X.

(b) Determine a reasonably simple expression for $\mathbb{E}[X]$, the mean of X.

Problem 2 (Sp19 Final). A random-number generator produces sample values of a continuous random variable U that is uniformly distributed between 0 and 1. In this problem you'll explore a method that uses the generated values of U to produce another random variable X that follows a desired probability law distinct from the uniform.

(a) Let $g : \mathbb{R} \to [0, 1]$ be a function that satisfies all the properties of a CDF. Furthermore, assume that g is invertible, i.e. for every $y \in (0, 1)$ there exists a unique $x \in \mathbb{R}$ such that $g(x) = y$ Let random variable X be given by $X = g^{-1}(U)$, where g^{-1} denotes the inverse of g. Prove that the CDF of X is $F_X(x) = g(x)$.

(b) A random variable X follows a double-exponential PDF given by

$$\forall x \in \mathbb{R}, \quad f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$

where $\lambda > 0$ is a fixed parameter. Using the random-number generator described above (which samples U), we want to generate sample values of X. Derive the explicit function that expresses X in terms of U. In other words, determine the expression on the right-hand side of

$$X = g^{-1}(U).$$

To do this, you must first determine the function g. From part (a) you know that $g(x) = F_X(x)$, so you must first determine $F_X(x)$.

Problem 3 (Fa18 Final). Suppose $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ are independent random variables, where $\lambda, \mu > 0$. What is $\mathbb{P}[\min\{X, Y\} \le t]$, where t is a positive constant?

Problem 4 (Fa18 Final). Suppose A and B are independent $\mathcal{N}(1, 1)$ random variables. Find $\mathbb{P}[2A+B \ge 4]$ in terms of the cumulative distribution function (c.d.f.) Φ of the standard normal distribution.

2 Solutions

Solution 1.

(a) For $x < 0$, the CDF of X is

$$F_X(x) = \mathbb{P}[X \leq x] = 0.$$

For $x \geq 0$ the CDF is given by

$$\begin{aligned} F_X(x) &= \mathbb{P}(-x \leq Z \leq x) \\ &= \Phi(x) - \Phi(-x) \\ &= \Phi(x) - \underbrace{[1 - \Phi(x)]}_{\Phi(-x)} \\ &= 2\Phi(x) - 1. \end{aligned}$$

Differentiating gives

$$f_X(x) = \frac{d}{dx} F_X(x) = 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \sqrt{\frac{2}{\pi}} e^{-z^2/2}.$$

(b) The mean of X is given by

$$\begin{aligned} \mathbb{E}(X) &= \int_0^\infty x f_X(x) dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2/2} dx \end{aligned}$$

Let $s = x^2/2$, so that $ds = xdx$. We then have

$$\mathbb{E}(X) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-s} ds$$

which leads to

$$\mathbb{E}(X) = \sqrt{\frac{2}{\pi}}.$$

Solution 2.

(a) The CDF of X is

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left[g^{-1}(U) \leq x\right]$$

since g is strictly increasing, we know that $g^{-1}(U) \leq x$ if, and only if,

$$g\left[g^{-1}(U)\right] = U \leq g(x).$$

Therefore,

$$F_X(x) = \mathbb{P}[U \leq g(x)] = F_U[g(x)] = g(x).$$

If g is differentiable, we can obtain the PDF of X as well:

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{dg(x)}{dx}.$$

Remark. If we want to simulate a random variable X that obeys a desired CDF $F_X(x)$, which is invertible over a range $S = \{x \mid 0 < g(x) < 1\}$ of interest, we can generate random variable U uniformly distributed in $[0,1)$, and let $X = F_X^{-1}(U)$.

(b) For $x < 0$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \frac{\lambda}{2}\int_{-\infty}^{x} e^{\lambda t}dt = \frac{1}{2}e^{\lambda x}.$$

For $x \geq 0$

$$F_X(x) = F_X(0) + \int_{0}^{x} f_X(t)dt = \frac{1}{2} + \frac{\lambda}{2}\int_{0}^{x} e^{-\lambda t}dt = \frac{1}{2} + \frac{1}{2}\left[1 - e^{-\lambda x}\right] = 1 - \frac{1}{2}e^{-\lambda x}.$$

That is,

$$g(x) = F_X(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & \text{if } x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & \text{if } x \geq 0. \end{cases}$$

To determine $X = g^{-1}(U)$, we consider two ranges of U separately: $0 \leq U < 1/2$ and $1/2 \leq U < 1$ We do this because for each of these two ranges the CDF $F_X(x)$ takes on distinct functional forms. If $0 \leq U < 1/2$, we let $F_X(X) = \frac{1}{2}e^{\lambda X} = U$. Solving for X, we obtain

$$X = \frac{1}{\lambda}\ln(2U).$$

If $1/2 \leq U < 1$, we let $F_X(X) = 1 - \frac{1}{2}e^{-\lambda X} = U$. Solving for X, we obtain

$$X = -\frac{1}{\lambda}\ln[2(1-U)].$$

Accordingly, we generate sample values of X as follows:

$$X = \begin{cases} \frac{1}{\lambda}\ln(2U) & \text{if } 0 \leq U < 1/2 \\ -\frac{1}{\lambda}\ln[2(1-U)] & \text{if } 1/2 \leq U < 1. \end{cases}$$

Solution 3. $1 - e^{-t(\lambda+\mu)}$.
$\mathbb{P}[\min\{X,Y\} \leq t] = 1 - \mathbb{P}[\min\{X,Y\} > t] = 1 - \mathbb{P}[X > t]\mathbb{P}[Y > t]$ because X and Y are independent. Thus our solution is $1 - e^{-t\lambda}e^{-t\mu} = 1 - e^{-t(\lambda+\mu)}$.

Solution 4. $1 - \Phi\left(\frac{1}{\sqrt{5}}\right)$.
Let $Z = 2A + B$. Then $\mathbb{E}[Z] = 2\mathbb{E}[A] + \mathbb{E}[B] = 3$ and $\mathrm{Var}[Z] = 4\,\mathrm{Var}[A] + \mathrm{Var}[B] = 5$ so Z is a Normal (3,5) random variable, and hence $\frac{Z-3}{\sqrt{5}}$ is standard normal. Then

$$\mathbb{P}[Z \geq 4] = \mathbb{P}[Z - 3 \geq 1] = \mathbb{P}\left[\frac{Z-3}{\sqrt{5}} \geq \frac{1}{\sqrt{5}}\right] = 1 - \Phi\left(\frac{1}{\sqrt{5}}\right).$$