Counting

Kelvin Lee

October 7, 2020

1 Rules of Counting

Theorem 1 (First Rule of Counting)**.** If there are n ways of doing something, and m ways of doing another thing after that, then there are $n \times m$ ways to perform both of these actions.

- Order matters(permutations).
- Sampling k elements from n items:
  - With replacement: n^k.
  - Without replacement: $\frac{n!}{(n-k)!}$.

Theorem 2 (Second Rule of Counting)**.** If order doesn't matter count ordered objects and then divide by number of orderings.

- Without replacement and ordering doesn't matter (combinations).
- Number of ways of choosing k-element subsets out of a set of size n:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

2 Stars and Bars

Stars and Bars is a technique used to solve for problems that sample with replacement but order doesn't matter by establishing a bijection between the problem and the stars and bars problem.

Problem 1. Consider the equation $a+b+c+d = 12$ where a, b, c, d are non-negative integers. How many solutions are there to this equation?

- Let's simplify this problem a little bit. Suppose we have 12 and 3 bars.

$$\star\star \mid \star\star \mid \star\star\star \mid \star\star\star\star\star$$

- How many ways can we arrange them? $\binom{12+3}{3} = \binom{15}{3}$
- This is the answer to our original problem! Do you see the bijection between the two problems?

Theorem 3 (Stars and Bars)**.** The number of ways to distribute n indistinguishable objects into k distinguishable bins is

$$\binom{n+k-1}{k-1}.$$

- Don't memorize the formula! Try to visualize the problem by connecting it to stars and bars. Draw out the stars and the bars!
- Again, this method is useful for with replacement but order doesn't matter type of problems.

Theorem 4 (Zeroth Rule of Counting:)**.** If a set A has a bijection relationship with a set B, then $|A| = |B|$.

The stars and bars method relies on this counting rule and this is the key to many combinatorial arguments as we will explore further later.

3 Binomial Theorem

Theorem 5 (Binomial Theorem)**.** For all $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. See notes. □

Corollary 6. For all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof. Plug in $a = -1$ and $b = 1$ for the binomial theorem. □

4 Combinatorial Proofs

- Intuitive counting arguments. No tedious algebraic manipulation.
- Proofs by stories: same story from multiple perspectives.
- Proving an identity by counting the same thing in two different ways.
- Useful identity:

$$\binom{n}{k} = \binom{n}{n-k}.$$

- Choosing k objects to include is equivalent to choosing $n-k$ objects to exclude.

Example 4.1. Using combinatorial arguments, show that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Proof. We can use binomial theorem by letting $a = b = 1$, however this is not what the question is asking for.
RHS: Total number of subsets of a set of size n.
LHS: The number of ways to choose a subset of size i is $\binom{n}{i}$. To find the total number of subsets, we simply add all the cases when $i = 0, 1, 2, \ldots, n$. □

5 Principle of Inclusion-Exclusion

Theorem 7 (Principle of Inclusion-Exclusion(General):)**.** Let $A_1, \ldots, A_n$ be arbitrary subsets of the same finite set A. Then,

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\}:|S|=k} \left|\cap_{i \in S} A_i\right|.$$

Proof. See notes. □

Theorem 8 (Principle of Inclusion-Exclusion(Simplified):)**.**

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

6 Summary

	with replacement	w/o replacement
order matters	n^k	$\frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k-1}$	$\binom{n}{k}$

7 Tips:

- Don't memorize formulas. Understand them by counting.
- Identify which categories does the problem fall under.
- Double check answers by using two different counting approaches.
- Check for overcounting.
- Relax and have fun!

8 Problems

Exercise 8.1. A mission to Mars will consist of 4 astronauts selected from 14 available. Exactly 5 of the 14 are trained in exobiology. If the mission requires at least 2 trained in exobiology, how many different crews can be selected?

Exercise 8.2. A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

Exercise 8.3. Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta, and Gamma choose their chairs?

Exercise 8.4 (Challenging)**.** Find

$$\sum_{k=0}^{49}(-1)^k\binom{99}{2k}.$$

Exercise 8.5 (AMC)**.** At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur within the group?