Counting and Probability Practice

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1 Problems

Problem 1. How many integer solutions does the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

have, if $x_1 \geq 2, x_2 \geq 3, x_3 \geq 10$ and $x_4 \geq -3$?

Problem 2. How many integer solutions are there to the system of inequalities

$$x_1 + x_2 + x_3 + x_4 \leq 15, \quad x_1, \ldots, x_4 \geq 0?$$

Problem 3. Count the number of non-negative integer solutions to

$$3x_1 + 3x_2 + 3x_3 + 7x_4 = 22.$$

Problem 4. Compute the number of injections $f : A \to B$ if $|A| = n$ and $|B| = m$.

Problem 5. There are five people of different height. In how many ways can they stand in a line, so there is no 3 consecutive people with increasing height.

Problem 6. For a fixed $1 \leq k \leq n$, what is the probability that a permutation σ of 1 through n satisfies the property that for all $i < k, \sigma(i) < \sigma(k)$? Express your answers in terms of n and k. Use this to compute the number of such permutations.

Problem 7. What is the probability that a permutation from 1 through n satisfies the property that for each $i, \sigma(\sigma(i)) = i$ and $\sigma(i) \neq i$? (For example, the permutation $3, 4, 1, 2$ is such a permutation, since for example $\sigma(\sigma(1)) = \sigma(3) = 1$. You may assume n is even.)

2 Answers

Answer 1. $\binom{3+4-1}{4-1}$.

Answer 2. $\binom{15+5-1}{5-1}$.

Answer 3. 10.

Answer 4. $\frac{m!}{(m-n)!}$

Answer 5. $5! - 3\binom{5}{3} \cdot 2! + \left[2\binom{5}{4} + 1\right] - 1$.

Answer 6. $\frac{1}{k}$.

Answer 7. $\frac{1}{2^{n/2}\left(\frac{n}{2}\right)!}$.