Counting Practice

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1 Problems

Exercise 1. How many ways can we distribute 10 candies to 5 kids so that each kid receives at least one?

Exercise 2 (Fa16 MT2)**.** How many combinations of even natural numbers (including zero) (x_1, x_2, x_3, x_4) are there such that $x_1 + x_2 + x_3 + x_4 = 20$?

Exercise 3 (Fa18 Final)**.** A 5-card poker hand is called a straight if its cards can be rearranged to form a contiguous sequence, regardless of their suits, i.e., if the hand is of the form $\{A, 2, 3, 4, 5\}, \{2, 3, 4, 5, 6\}, \dots,$ or $\{10, J, Q, K, A\}$. How many straight hands are there consisting of 3 black and 2 red cards?

Exercise 4 (Sp15 MT2)**.** How many different ways are there to rearrange the letters of DIAGONALIZATION without the two N's being adjacent?

Exercise 5 (Sp15 Final)**.** What is the number of ways of placing k labelled balls in n labelled bins such that no two balls are in the same bin? Assume $k \leq n$.

Exercise 6 (Sp16 Final)**.** Compute the number of ways to split n dollars among k people where at most one gets zero dollars.

Exercise 7 (Sp15 MT2)**.** How many non-decreasing sequences of k numbers from $\{1, ..., n\}$ are there? For example, for $n = 12$ and $k = 7$, $(2, 3, 3, 6, 9, 9, 12)$ is a non-decreasing sequence, but $(2, 3, 3, 9, 9, 6, 12)$ is not.

Exercise 8 (Sp19 Final)**.** How many $(x_1, \dots, x_k, y_1, y_2, \dots, y_k)$ are there such that all x_i, y_i are non-negative integers, $\sum_{i=1}^{k} x_i = n$, and $y_i \leq x_i$ for $1 \leq i \leq k$? Answer may *not* include any summations.

2 Solutions

Solution 1. Reserve one candy for each kid, so we remove 5 candies and are left with 5 remaining. Then we directly use stars and bars and get $\binom{5+5-1}{5-1} = \boxed{\binom{9}{4}}$.

Solution 2. Even numbers have the form $2k$, so if divide the equation by 2, we get $x_1' + x_2' + x_3' + x_4' = 10$. For every solution of this form, we can construct our desired solution (x_1, x_2, x_3, x_4) by multiplying by 2 (bijection). Then by stars and bars, we have $\binom{10+4-1}{4-1}) = \boxed{\binom{13}{3}}$.

Solution 3. There are 10 distinct sets of numeric values that can form a straight. Given such a set of five numbers, there are $\binom{5}{3}$ ways of choosing which ones are red and which ones are black, and given the color of a card, there are two different suits that share this colour. So $\boxed{10 \cdot \binom{5}{3} \cdot 2^5}$.

Solution 4. The word DIAGONALIZATION has 15 letters with 3 A's, 3 I's, 2 N 's, and 2 O's, so there are $\frac{15!}{3!3!2!2!}$ ways to rearrange the letters in total. The number of rearrangements where the two N's are adjacent is $\frac{14!}{3!3!2!}$, where we have considered "NN" as a single character. The difference $\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!}$ is then equal to the number of rearrangements without the two N's being adjacent. Hence, we get $\boxed{\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!}}$

Solution 5. There are $\binom{n}{k}$ ways to choose k bins to place those labelled balls. There are $k!$ ways to arrange those balls among the k assigned bins. Thus, the answer is $\boxed{\binom{n}{k} \cdot k!}$.

Solution 6. We first count the ways where exactly one gets zero; there are k choices for the person who may get zero, then we distribute the dollars to the $k-1$ remaining people so that each gets at least one. Then we add the ways for all to get at least one. So we have $\boxed{k\binom{n-(k-1)+(k-2)}{k-2} + \binom{n-(k-1)+(k-1)}{k-1} = k\binom{n-1}{k-2} + \binom{n-1}{k-1}}$.

Solution 7. Each non-decreasing sequence is specified by how many times each element $i \in \{1, \ldots, n\}$ appears in the sequence. Therefore, the number of non-decreasing sequences of length k is equal to the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_n = k$$

where $x_i \geq 0$ is the number of times i appears in the sequence. Therefore, the number of such non-decreasing sequences is $\boxed{\binom{k+n-1}{n-1}}$ by stars and bars.

Solution 8. Define $z_i = x_i - y_i$ for each i. Then, we see that $z_i \geq 0$ and

$$\sum_{i=1}^{k} z_i + \sum_{i=1}^{k} y_i = n.$$

From stars and bars, there are $\boxed{\binom{n+2k-1}{2k-1}}$ ways to pick the y_i and the z_i. We can uniquely construct x_i from y_i and z_i, so this is our final answer.