Discrete Fourier Transform

- **Roots of Unity**

$$\omega = e^{\frac{2\pi}{N}nj}, \quad n = 0, \dots, N-1$$

$$\omega_N = e^{\frac{2\pi}{N}j}$$

$$\boxed{\sum_{n=0}^{N-1} \omega_N^n = 0}$$

- **Conjugate pair**

$$\omega^N = 1 \implies (\overline{\omega})^N = 1$$

- **DFT Basis**
  Define basis $\{u_0, \dots, u_{N-1}\}$ for $\mathbb{C}^N$ as:

$$u_k[n] = \frac{1}{\sqrt{N}} \omega_N^{kn}$$

Theorem 1 (DFT basis is orthonormal)**.**

Proof.

$$\begin{aligned}
\langle u_k, u_{k'} \rangle &= \sum_{n=0}^{N-1} u_k[n] \overline{u_{k'}[n]} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \omega_N^{kn} \omega_N^{-k'n} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \omega_N^{(k-k')n} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \zeta^n
\end{aligned}$$

□

Definition 1 (Synthesis)**.** *The change of coordinates from the DFT basis,* $F^* \in \mathbb{C}^{n\times n}$*.*

$$F^*_{kn} = \frac{1}{\sqrt{N}}\omega_N^{kn}$$

$$F^* = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ 1 & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \qquad \textbf{(Synthesis Matrix)}$$

$$\boxed{x = F^* X} \qquad \textbf{(Synthesis Equation)}$$

Definition 2 (Analysis)**.** *The change of coordinates to the DFT basis,* $F \in \mathbb{C}^{n\times n}$*.*

$$F_{kn} = \overline{F^*_{nk}} = \frac{1}{\sqrt{N}}\omega_N^{-kn}$$

$$F = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & \omega^{-(N-1)} \\ 1 & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \cdots & \omega^{-(N-1)(N-1)} \end{pmatrix} \qquad \textbf{(Analysis Matrix)}$$

$$\boxed{X = Fx} \qquad \textbf{(Analysis Equation)}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}} = \sum_{n=0}^{N-1} x[n]\omega_N^{-kn} \qquad \textbf{analysis}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} X[k]\omega_N^{kn} \qquad \textbf{synthesis}$$

$$\boxed{x[n] = \sum_{k=0}^{N-1} X[k]u_k[n]}$$

- **DFT of sinusoid**

$$\begin{aligned}
x[n] &= \alpha \cos\left(\frac{2\pi k}{N} n + \varphi\right), \quad n = 0, \ldots, N-1 \\
&= \frac{\alpha}{2}\left(e^{\left(\frac{2\pi k}{N} n + \varphi\right) j} + e^{-\left(\frac{2\pi k}{N} n + \varphi\right) j}\right) \\
&= \frac{\alpha}{2}\left(e^{\frac{2\pi k}{N} n j} e^{\varphi j} + e^{-\frac{2\pi k}{N} n j} e^{-\varphi j}\right) \\
&= \frac{\alpha}{2}\left(\left(e^{\frac{2\pi}{N} j}\right)^{kn} e^{\varphi j} + \left(e^{\frac{2\pi}{N} j}\right)^{-kn} e^{-\varphi j}\right) \\
&= \frac{\alpha}{2}\left(\omega_N^{kn} e^{\varphi j} + \omega_N^{-kn} e^{-\varphi j}\right) \\
&= \frac{\alpha}{2} e^{\varphi j} \omega_N^{kn} + \frac{\alpha}{2} e^{-\varphi j} \omega_N^{-kn} \\
&= \frac{\alpha\sqrt{N}}{2} e^{\varphi j} u_k[n] + \frac{\alpha\sqrt{N}}{2} e^{-\varphi j} u_{N-k}[n]
\end{aligned}$$

$$\boxed{x = \frac{\alpha\sqrt{N}}{2} e^{\varphi j} u_k + \frac{\alpha\sqrt{N}}{2} e^{-\varphi j} u_{N-k}}$$

If $k = 0$ modulo N, then $u_k = u_0 = u_{N-k}$:

$$x = \left(\alpha\sqrt{N}\cos\varphi\right) u_0$$

else:

$$X[n] = \begin{cases} \frac{\alpha\sqrt{N}}{2} e^{\varphi j}, & n = k \\ \frac{\alpha\sqrt{N}}{2} e^{-\varphi j}, & n = N-k \\ 0, & \text{else} \end{cases}$$

- **Properties of DFT:**
  - Linear:

$$F(ax + by) = aFx + bFy = aX + bY$$

  - Energy-preserving (**Parseval's Theorem**):

$$\|Fx\|^2 = \|x\|^2$$

  - Conjugate-symmetric for real signals: if x is **real**, then

$$X[n] = \overline{X[-n]} = \overline{X[N-n]}$$

DFT of A Square Wave

- Let $x \in \mathbb{C}^N$ be the following rectangular pulse, which approximates a square wave when $M = N/4$

$$x[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{else} \end{cases}$$

$$\boxed{X[n] \approx \frac{\sqrt{N}}{2}\operatorname{sinc}\left(\frac{1}{2}n\right)}, \quad n = -\left\lceil\frac{N}{2}\right\rceil \dots 0 \dots \left\lceil\frac{N}{2}\right\rceil$$

- **"The DFT of a square is a sinc and the DFT of a sinc is a square."**
- If square wave x and sinc X:

$$X = Fx$$

If both x and X are real,

$$\overline{X} = \overline{Fx} \implies X = \overline{F}x = F^*x$$

$$\boxed{FX = x}$$

*Only true when x and X are both **real** and $F \neq F^*$!*

Linear Time Invariant (LIT) Systems

- **Properties:**
  - Linear:
    1. Scaling:
       If $u[n] \to y[n]$ and $u_2[n]$, then
       $$au[n] \to ay[n]$$
    2. Superposition:
       If $u_1[n] \to y_1[n]$ and $u_2[n] \to y_2[n]$, then
       $$u_1[n] + u_2[n] \to y_1[n] + y_2[n]$$
  - Time Invariant:
    $$u[n - n_0] \to y[n - n_0]$$
- Impulse response

$$\delta[n] \triangleq \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{y[n] = \sum_m u[m]h[n-m]}$$