CS70 Final Review (Concentration Inequalities)

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December 8, 2020

Let's say that for some real $b \geq 1$ that a random variable X is *b-reasonable* if it satisfies:

$$\mathbb{E}[X^4] \leq b\left(\mathbb{E}[X^2]\right)^2.$$

Suppose X is *b-reasonable* and that $\mathbb{E}[X] = 0$ and $\mathrm{Var}(X) = \sigma^2$. Our goal is to prove that

$$\mathbb{P}[|X| \geq t\sigma] \leq \frac{b}{t^4}.$$

(a) Find an expression for σ in terms of $\mathbb{E}[X^2]$.

(b) Show that $\mathbb{P}[|X| \geq t\sigma] \leq \frac{b}{t^4}$. (**Hint:** find an equivalent event for $|X| \geq t\sigma$.)

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