Random Variables Practice

Kelvin Lee

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1 Problems

Problem 1 (Sp19 Final)**.** Given independent $X, Y \sim \text{Bin}(n, p)$, what is $P(X + Y = i)$?

Problem 2 (Sp18 Final)**.** Given a binomial random variable X with parameters n and p, ($X \sim B(n, p)$) what is $P(X = \mathbb{E}[X])$?

Problem 3 (Sp19 Final)**.** Consider throwing n balls into n bins uniformly at random. Let X be the number of balls in the first bin. What is $\mathbb{E}[X]$ and $\text{Var}(X)$?

Problem 4 (Sp19 Final)**.** Now let Y be the number of balls in the second bin. What the joint distribution of X, Y, i.e., what is $P(X = i, Y = j)$? What is $P(X = i \mid Y = j)$?

Problem 5 (Fa18 Final)**.** Consider an urn with 3 blue balls and 1 red ball, and suppose you sample one ball at a time with replacement. Let X be the number of draws required until both of the colors, blue and red, have been observed at least once.

(i) What is $P(X = n \mid \text{The first ball drawn is red })$, for $n \geq 2$?

(ii) What is $P(X = n)$, for $n \geq 2$?

Problem 6 (Sp18 Final)**.** Consider $X \sim Geo(p)$, a geometric random variable X with parameter p. What is $P(X > i \mid X > j)$ for $i \geq j$?

2 Answers

Answer 1. $\binom{2n}{i}p^i(1-p)^{2n-i}.X + Y \sim \text{Bin}(2n, p)$.

Answer 2. $\binom{n}{np}p^{np}(1-p)^{n-np}$

Answer 3. One can use linearity of expectation: $X_1 + \ldots + X_n$, where X_1 indicates that ball i falls into bin 1 and $\mathbb{E}[X_1] = \frac{1}{n}$. $\text{Var}(X) = 1 - \frac{1}{n}$. Take $X = X_1 + \ldots + X_n$ where X_i is an indicator random variable for choosing ball i choosing bin 1. From the fact that each X_i has variance $\frac{1}{n}\left(1 - \frac{1}{n}\right)$ and the fact that X_i s are independent from each other, we get $n\left(\frac{1}{n}\left(1 - \frac{1}{n}\right)\right) = (1 - 1/n)$

Answer 4. $\binom{n}{i}\binom{n-i}{j}\left(\frac{1}{n}\right)^{i+j}\left(1 - \frac{2}{n}\right)^{n-i-j}$. There are $\binom{n}{i}$ ways to choose which balls will go into the first bin, and $\binom{n-i}{j}$ ways to pick which balls go into the second bin. After picking, the probability that the i balls actually end up in the first bin is $\frac{1}{n^i}$, the probability that the j balls end up in the second bin is $\frac{1}{n^j}$ and the probability that the other $n - i - j$ balls end up in not the first and second bins is $\left(1 - \frac{2}{n}\right)^{n-i-j}$.
$\binom{n-j}{i}\left(\frac{1}{n-1}\right)^i\left(\frac{n-2}{n-1}\right)^{n-i-j}$. We can pretend the second bin doesn't exist, since it will have j balls already. Then, there are $n - j$ balls left, thrown into $n - 1$ bins.

Answer 5.

(i) $\left(\frac{1}{4}\right)^{n-2}\frac{3}{4}$. Given that we first drew a red ball, if we take n draws to observe both colors, then draws 2 to $n-1$ must also have been red and draw n must be blue. This occurs with probability $\left(\frac{1}{4}\right)^{n-2}\frac{3}{4}$. (Observe that we are following a geometric distribution)

(ii) $\frac{3+3^{n-1}}{4^n}$. Let R denote the event that the first ball drawn is red and B denote the event that the first ball drawn is blue. Using a similar argument as in (i) we have $P(X = n \mid B) = \left(\frac{3}{4}\right)^{n-2}\frac{1}{4}$ Thus

$$\begin{aligned} P(X = n) &= P(X = n \mid R) \cdot P(R) + P(X = n \mid B) \cdot P(B) \\ &= \left(\frac{1}{4}\right)^{n-1} \cdot \frac{3}{4} + \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} = \frac{3 + 3^{n-1}}{4^n} \end{aligned}$$

Answer 6. $(1-p)^{i-j}$. This is the event that the first $i - j$ trials fail after the j th trial.